Introduction to Stellarators

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Will give two talks at the workshop and four lectures after the workshop.

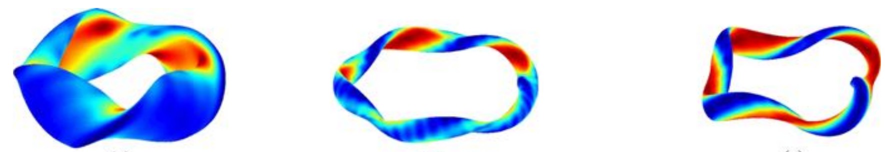
1. Introduction to Stellarators (This talk)
2. Stellarator Divertors
3. Large Stellarator Experiments and Reactors
4. Mathematical Description of Stellarators
5. Stellarator Optimization
6. Planning Stellarator Research

These talks are intended to be sufficiently independent that later talks can be understood without having heard the earlier talks.

Topics to be Discussed

Why a poloidal field is required.

Ways the poloidal field can be produced.



(a) Quasi-Axisymmetric NCSX

(b) Quasi-Isodynamic W7-X

(c) Quasi-Helically Symmetric HSX

Benefits of poloidal field production by external coils.

Poloidal field production.

Particle confinement in stellarators.

Important adiabatic invariants.

Controlling magnetic field strength to be quasi-symmetric.

Controlling magnetic field strength to be quasi-isodynamic.

Three types of stellarators that give enhanced particle confinement.

Poloidal Magnetic Field Required for Toroidal Plasma Confinement

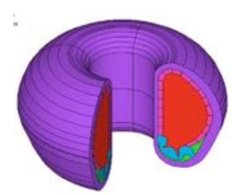
Both a poloidal (short way) and a toroidal (long way) magnetic field are required to balance the plasma pressure gradient $\vec{\nabla}p = \vec{j} \times \vec{B}$.

Follows from $\vec{\nabla} \cdot \vec{j} = 0$ in an equilibirium and $\vec{j} = (j_{||}/B)\vec{B} + \vec{j}_\perp$, so

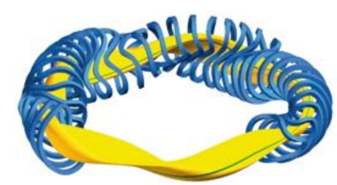
$$\vec{B} \cdot \vec{\nabla}\frac{j_{||}}{B} = -\vec{\nabla} \cdot \vec{j}_\perp = -(\vec{B} \times \vec{\nabla}p) \cdot \vec{\nabla}\frac{1}{B^2} = \frac{2}{B}\frac{\partial p}{\partial z} \quad \text{when} \quad \vec{B} = \frac{\mu_0 G}{2\pi R}\hat{\varphi}.$$

The return current, $\vec{B} \cdot \vec{\nabla}(j_{||}/B)$, requires poloidal field to balance $\partial p/\partial z$.

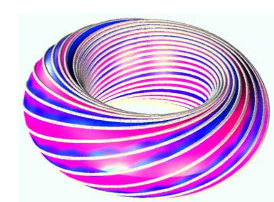
Two toroidal concepts have been explored at a large scale for the confinement of fusion plasmas by magnetic fields: tokamaks and stellarators.



ITER Tokamak



W7-X Stellarator

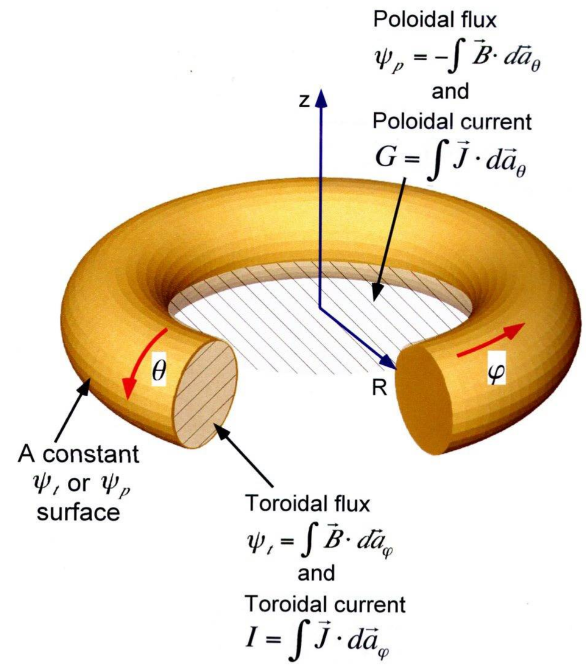


Magnetic Field

Ways to Produce the Poloidal Magnetic Field

The poloidal magnetic field can be produced in three ways

1. A net plasma current I (only possibility in an axisymmetric tokamak).
2. A helical magnetic axis.
3. Shaped (often elliptical) magnetic surfaces (constant p surfaces) that rotate with φ, the toroidal angle.



A helical magnetic axis and rotating elliptical surfaces can be produced by external coils.

Stellarators by definition have a large part of their poloidal magnetic field produced by external coils.

Stellarators cannot be axisymmetric, which makes the concept more subtle and the design more difficult.

Importance of Poloidal Field Production by External Coils

When the poloidal field is produced by a net plasma current I:

1. The maintenance of I is a major power drain. The maintenance of an electron current against drag on background electrons requires a power density $\gtrsim j_{net}E_{ch}$. Bootstrap current reduces but doesn't solve this problem. E_{ch} is the Connor-Hastie electric field; $|eE_{ch}|$ is the minimum force from background plasma drag for an electron of arbitrary energy.

2. The current profile $j(r)$ is sensitive to non-linear transport physics and, therefore, difficult to control.

3. The net current can be transferred from near thermal to relativistic electron carriers, which could do major machine damage. This issue has not been adequately addressed for ITER and is more severe in a tokamak reactor, NF **58**, 036006 (2018).

A strong externally-produced poloidal field can:

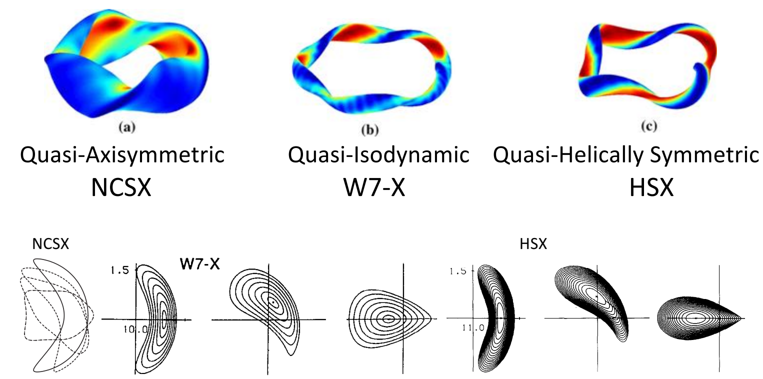
1. Center the plasma in the chamber making disruptions avoidable.

2. As illustrated by W7-X, computational design allows large steps to be taken for fast stellarator development—which is not possible for tokamaks.

Poloidal Field Production

External coils can produce the toroidal field, when G_c is the net poloidal current in the coils,

$$B_\varphi = \frac{\mu_0 G_c}{2\pi R},$$



Large Aspect Ratio Approximation

The physics is easier to explain when the aspect ratio of the torus $R_0/a \to \infty$. One can then use cylindrical coordinates (r, θ, z), where $z = R_0\varphi$, the periodicity distance around the torus is $2\pi R_0$, and the toroidal angle is φ. For tokamaks and stellarators, $B_\varphi/B_\theta \propto R_0/a \to \infty$.

Positions in space are given by (Note θ has a backward sign from usual to make $(\hat{r} \times \hat{\theta}) \cdot \hat{\varphi} > 0$ with $\hat{\varphi} = \hat{z}$.)

$$\vec{x}(r, \theta, \varphi) = r\hat{r}(\theta) + R_0\varphi\hat{z}, \text{ where } \qquad \frac{d\hat{r}}{d\theta} = -\hat{\theta} \text{ and } \frac{d\hat{\theta}}{d\theta} = \hat{r}.$$

Poloidal Field due to a Net Plasma Current I

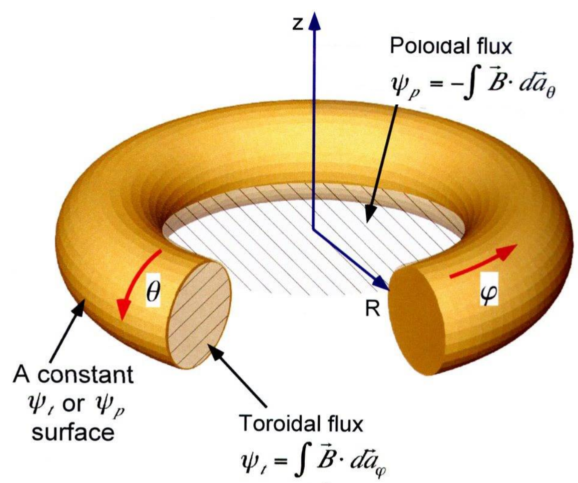
$$B_\theta = \frac{\mu_0 I(r)}{2\pi r}; \quad \text{field line twist is} \quad \iota(r) = \frac{d\theta}{d\varphi} = \frac{\vec{B}\cdot\vec{\nabla}\theta}{\vec{B}\cdot\vec{\nabla}\varphi} = \frac{B_\theta/r}{B_\varphi/R} = \frac{R}{r}\frac{B_\theta}{B_\varphi}.$$

$\iota(r)$ is called the rotational transform and is the inverse of the safety factor $\iota(r) = 1/q(r)$. In tokamaks, $q(0) \approx 1$ and at the edge $r = a$ the safety factor $q(a) \approx 3$.

The poloidal magnetic flux is $\psi_p = 2\pi R_0 \int B_\theta dr$.

The toroidal magnetic flux $\psi_t = \int B_\varphi 2\pi r dr$.

The rotational transform $\iota \equiv \dfrac{d\psi_p}{d\psi_t} = \dfrac{R}{r}\dfrac{B_\theta}{B_\varphi}$.



Poloidal Field due to a Helical Magnetic Axis

Circular, $r = r_0$ magnetic surfaces around an arbitrary magnetic axis, $\vec{x}_a(\ell)$, are given by

$$\vec{x}(r,\vartheta,\ell) = \vec{x}_a(\ell) + r\cos\vartheta\hat{\kappa} + r\sin\vartheta\hat{\tau}.$$

The orthonormal Frenet unit vectors are $\hat{b}_a = d\vec{x}_a/d\ell$, $\hat{\kappa}$, and $\hat{\tau}$, where $d\hat{b}_a/d\ell = \kappa\hat{\kappa}$ with $\kappa\hat{\kappa}$ the curvature of the axis, $d\hat{\kappa}/d\ell = -(\kappa\hat{b}_a + \tau\hat{\tau})$ with $\tau\hat{\tau}$ the torsion of the axis, $d\hat{\tau}/d\ell = \tau\hat{\kappa}$, and $\hat{b}_a \times \hat{\kappa} = \hat{\tau}$.

$$\frac{d\vec{x}}{d\ell} = \hat{b}_a + r\sin\theta\left(\tau - \frac{d\vartheta}{d\ell}\right)\hat{\kappa} + r\cos\theta\left(\frac{d\vartheta}{d\ell} - \tau\right)\hat{\tau}$$

The rotational transform due to torsion ι_t can be found from the advance in ϑ per toriodal circuit,

$$\Delta\vartheta = \oint \tau d\ell = 2\pi\left(\iota_\tau - N_p\right)$$

N_p is the number of times the curvature vector winds around the axis, where

$$\vec{x}_a = \frac{R_0\Delta_\tau}{N_p}\hat{\rho}(\varphi) + R_0\hat{R}(\varphi), \text{ with } \hat{\rho} \equiv \hat{R}\cos N_p\varphi - \hat{Z}\sin N_p\varphi.$$

$$\iota_\tau = N_p\left(1 - \frac{1}{\sqrt{1+\Delta_\tau^2}}\right) \approx N_p\frac{\Delta_\tau^2}{2}.$$

Poloidal Field due to a Helically Shaped Magnetic Surfaces

Magnetic surfaces have helical shaping if the minor radius can be written as

$$r = r_0\left\{1 - \Delta_{MN}\sin(M\theta - N_p\varphi)\right\},$$

where the average minor radius is r_0, the poloidal angle is θ, the toroidal angle is φ, and N_p is the number of periods.

For a curl-free magnetic field, $\vec{B} = \vec{\nabla}\phi$, where $\nabla^2\phi = 0$, and

$$\phi = \frac{N_p r_0}{M R_0} r_0 \Delta_{MN} B_0 \left(\frac{r}{r_0}\right)^M \cos(M\theta - N_p\varphi), \text{ which implies } \Delta_{MN} \propto r_0^{(m-2)}.$$

When the inverse aspect ratio period is small, $\epsilon_p \equiv N_p r_0/R_0 << 1$, and the shaping is weak $|\Delta_{mn}| << 1$, the rotational transform due to helically shaped surfaces

$$\iota_h = N_p(M-1)\Delta_{MN}^2, \text{ with } \Delta_{MN} \propto r_0^{m-2} \text{ determining } r_0 \text{ dependence of } \iota_h\text{(r)}.$$

The variation in field strength associated with the shaping is $\delta B_{MN}/B_0 = (N_p r_0/R_0)\Delta_{MN}$.

For details of derivations of ι_τ and ι_h, see p.37 of Boozer, Nucl. Fusion **55**, 25001 (2015).

Particle Confinement in Stellarators

Magnetic fusion systems are in an odd collisionality regime.

Particles are collisional in the sense that the required energy confinement time τ_E is much longer that the collision time τ_c with $\tau_E/\tau_c \approx 10^2$ for ions and $\tau_E/\tau_c \approx 10^4$ for electrons. The root-mean-square deviation from a Mawellian distribution $\Delta f_{rms}/f_{Max} \approx \sqrt{\tau_c/\tau_E}$. Writing $f = f_{Max}e^{\hat{f}}$, the entropy production $\dot{s} = -\int \hat{f}C(f)d^3v = \nu_c n < \hat{f}^2 >$.

Particles are collisionless in the sense that the mean-free-path $\tau_c\sqrt{T/m} >> R_0$.

The implication is that all collisionless particle trajectories in the Maxwellian must be well confined.

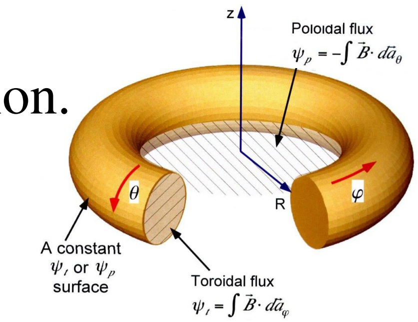
The confinement of particles in a stellarator depends on adiabatic invariants—just as the confinement of planets in the solar system does.

In Hamiltonian mechanics, $H(p,q,t)$, when the motion is oscillatory with a frequency ω and changes little from one oscillation to the next, the action

$$J \equiv \oint pdq \text{ is an adiabatic ivariant.}$$

The integral is over an oscillation and J changes by an exponentially small amount $\exp(-1/\omega\tau_{ev})$. The time scale over which the oscillatory motion changes is τ_{ev}.

Important Adiabatic Invariants



Magnetic moment $\boldsymbol{\mu \equiv \dfrac{mv_\perp^2}{2B}}$, given by the gyrofrequency oscillation.

The energy $H = mv_{||}^2/2 + \mu B + q\Phi$, where the electric potential Φ is usually constant along the magnetic field. A particle is trapped and bounces back and forth when

$\mu B_{max} > \dfrac{mv_{||}^2}{2} + \mu B$; B_{max} is the maximum magnetic field strength along the field line.

Otherwise the particle is passing, and it moves along the magnetic field line forever.

Longitudinal action $\boldsymbol{J(H, \psi_t, \Theta) \equiv \oint mv_{||} d\ell}$, an integral over the bounce motion.

The magnetic field is written in the Clebsch representation $\vec{B} = \vec{\nabla}\psi_t \times \vec{\nabla}\Theta/2\pi$, where $\theta = \Theta + \iota\varphi$ and ℓ is the distance along a line.

A magnetic field is quasi-symmetric when $\partial J/\partial\Theta = 0$, as in an axisymmetric tokamak or in stellarator that has exact quasi-symmetry.

When the precession given by $|\partial J/\partial\psi_t|$ is sufficiently large, the radial drift (drift across enclosed toroidal flux) is small.

Controlling the Magnetic Field Strength for Quasi-symmetry

The confinement of passing particles, $mv_{||}^2 > \mu(B_{max} - B)$, is determined by the confinement of magnetic field lines—whether the lines form magnetic surfaces.

The confinement of trapped particles, $mv_{||}^2 < \mu(B_{max} - B)$, is determined by the longitudinal action J and hence the variation in the magnetic field strength along the magnetic field lines.

To have quasi-symmetry the magnetic field strength must be a constant B_a along the magnetic axis.

Near the magnetic axis, the magnetic field strength is given by

$$B(\psi_t,\theta,\varphi) = B_a(1+\vec{\kappa}\cdot\delta\vec{x}), \text{ where } \vec{x}(\psi,\theta,\varphi) = \vec{x}_a(\theta,\varphi)+\delta\vec{x}(\psi_t,\theta,\varphi).$$

The vector identity for $\vec{\nabla}(\vec{B}\cdot\vec{B})$, the equation $\vec{\nabla}p = \vec{j}\times\vec{B}$, and the curvature $\vec{\kappa} \equiv \hat{b}\cdot\vec{\nabla}\hat{b}$ imply $\vec{\nabla}_\perp(B^2+2\mu_0 p) = 2B^2\vec{\kappa}$.

To lowest, $\sqrt{\psi_t} \propto r$, order, $\hat{\kappa}\cdot\delta\vec{x} = w(\varphi)\sqrt{\psi_t}\cos(\theta - N_B\varphi)$. Width of surface is $w(\varphi)$.

For $B \propto \cos(\theta - N_B\varphi)$ need $w(\varphi)\kappa(\varphi)$ constant.

Controlling the Magnetic Field Strength to be Quasi-isodynamic

The magnetic field strength can also be controlled to obtain good particle confinement through quasi-poloidal-symmetry, also known as quasi-isodynamic. Quasi-poloidally-symmetric fields can depend on θ unlike quasi-symmetric fields which depend on only $\theta - N_B\varphi$.

To have quasi-poloidal-symmetry, the magnetic field strength varies along the magnetic axis, $B_a(\varphi)$, forming a series of magnetic mirrors.
xis is given by

$$B(\psi_t, \theta, \varphi) = B_a(\varphi)(1 + \vec{\kappa} \cdot \delta\vec{x}), \text{ where } \vec{x}(\psi, \theta, \varphi) = \vec{x}_a(\theta, \varphi) + \delta\vec{x}(\psi_t, \theta, \varphi).$$

To lowest, $\sqrt{\psi_t} \propto r$, order, $\hat{\kappa} \cdot \delta\vec{x} = w(\varphi)\sqrt{\psi_t}\cos(\theta - N\varphi)$.

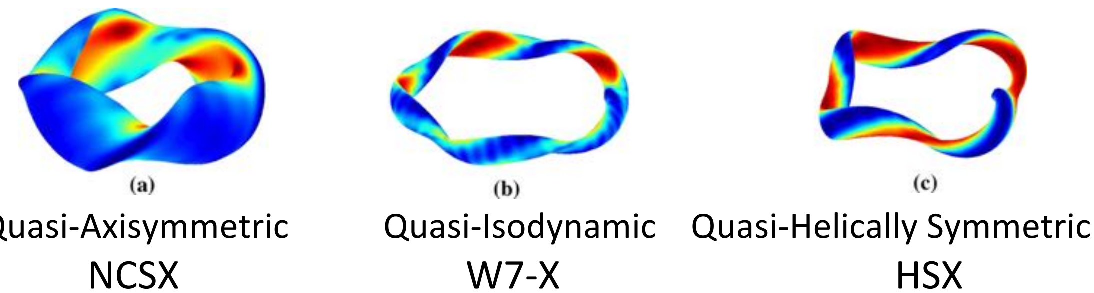
There is no radial drift when $(\vec{B} \times \vec{\nabla} B) \cdot \vec{\nabla}\psi_t = 0$. This condition is satisfied to $\sqrt{\psi_t}$ order when the curvature of the axis vanishes, $\vec{\kappa} = \vec{b}_a \cdot \vec{\nabla}\vec{b}_a = 0$.

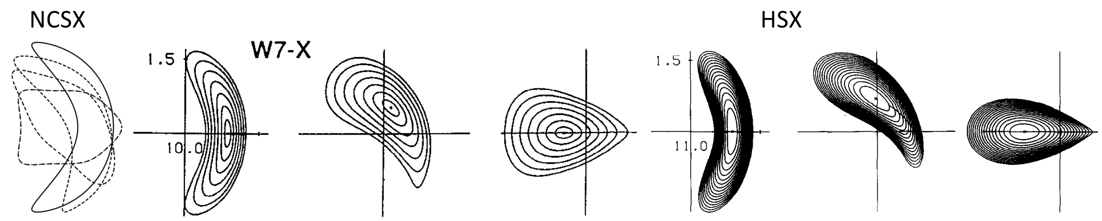
In a torus, the axis cannot have zero curvature everywhere. But as on W7-X, the regions of large curvature can be restricted to be near the maxima of the magnetic field strength, which few trapped particles sample, with $w(\varphi)$ small there. The radial drift, $\propto \partial J/\partial\alpha$, is then small and can be limited by the precession, $\propto \partial J/\partial\psi$.

See Helander, Rep. Prog. Phys. **77**, 087001 (2014) for an extensive discussion of quasi-isodynamic stellarators, pages 21-24.

Three Types of Stellarators for Enhanced Particle Confinement

Quasi-axisymmetric stellarators have the tightest aspect ratio. Have particle trajectories similar to those of an axisymmetric tokamak. Plasma surface has sharp edges which of finite length.





Quas-isodynamic stellarators are being studied at large size. Have the least change in the plasma state as the pressure is made larger. Are the only type of stellarator for which is easy to use edge islands for a divertor.

Quasi-helical stellarators appear to address all of the physics issues when the number of periods N_p is sufficiently large. The particle trajectories approach those of exact helical symmetry as $1/N_p^3$.

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Benefits of poloidal field production by external coils.

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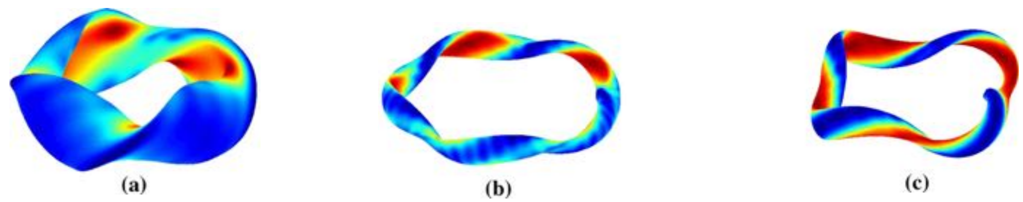
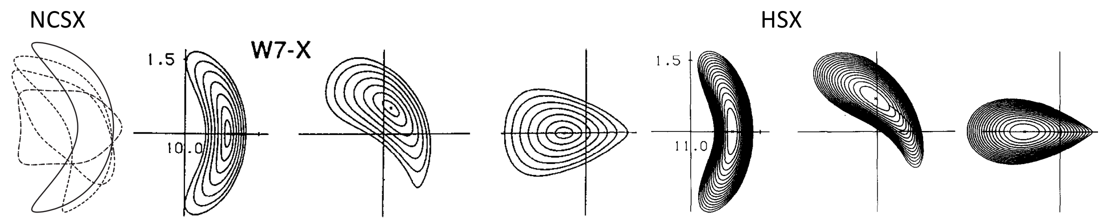


Quasi-Axisymmetric NCSX | Quasi-Isodynamic W7-X | Quasi-Helically Symmetric HSX



Important adiabatic invariants.

Controlling magnetic field strength to be quasi-symmetric.

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