Fully Nonlinear Gyrokinetic Equations for Magnetized Plasmas

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Abstract. Nonlinear electromagnetic gyrokinetic equations have been constructed without expanding the field variables into background and finite but small-amplitude fluctuating components. In the long-wavelength limit, these fully (un-expanded) nonlinear gyrokinetic equations recover the well-known drift-kinetic equations. In the expanded limit, they recover the usual nonlinear gyrokinetic equations. These equations, thus, can be applied toward long-time simulations covering from microscopic to macroscopic spatial scales.

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1. Introduction

In magnetized plasmas, reduced kinetic equations are often employed to study low-frequency electromagnetic fluctuations either analytically or by direct numerical simulations. Such reduced kinetic equations are derived via systematic expansions of the original complete kinetic equations in terms of the small ratio between the short gyroperiod and the long fluctuation time scales. Two well-known reduced kinetic equations are the drift-kinetic equation [1, 2], and the gyrokinetic equation [3, 4, 5, 6, 7, 8].

More specifically, let $\epsilon \equiv \rho/L$ be the small expansion parameter. Here, $\rho = v_t/\Omega$ and L are, respectively, the Larmor radius and the scale length of the magnetic field $\mathbf{B}$; v_t is the characteristic (thermal) speed, and Ω is the cyclotron frequency. Furthermore, since the electromagnetic fields consist, typically, of background and fluctuating components, let us take ω and $\mathbf{k} = \mathbf{k}_\perp + k_\parallel \mathbf{b}$ be, respectively, the characteristic frequency and wave vector of the electromagnetic fluctuations. Here, $\mathbf{b} = \mathbf{B}/B$ is the unit vector along the field line, $\perp$ and $\parallel$ denote the components perpendicular and parallel to $\mathbf{B}$, respectively. The drift-kinetic orderings and the corresponding drift-kinetic equation are then [1]

$$|\frac{\omega}{\Omega}| \sim |k\rho| \sim \mathcal{O}(\epsilon), \tag{1}$$

and

$$[\partial_t + \dot{\mathbf{R}} \cdot \nabla + \dot{V}_\parallel \partial_{V_\parallel}] F_d(\mathbf{R}, V_\parallel, \mu, t) = 0. \tag{2}$$

Here, $\mathbf{R}$ denotes the guiding-center position, and the guiding-center phase space motion ($\dot{\mathbf{R}}$, $\dot{V}_\parallel$) will be described by the guiding-center velocity

$$\dot{\mathbf{R}} = V_\parallel \mathbf{b} + \mathbf{V}_E + \mathbf{V}_d, \tag{3}$$

with

$$\mathbf{V}_E = \frac{c\mathbf{E} \times \mathbf{b}}{B}, \tag{4}$$

$$\mathbf{V}_d = \frac{1}{\Omega} \mathbf{b} \times (\mu \nabla B + V_\parallel^2 \kappa), \tag{5}$$

and the parallel force equation

$$\dot{V}_\parallel = \frac{\mathbf{B}^*}{B_\parallel^*} \cdot (\frac{q}{m}\mathbf{E} - \mu \nabla B), \tag{6}$$

with the modified magnetic field

$$\mathbf{B}^* = \mathbf{B} + \frac{V_\parallel B}{\Omega} \nabla \times \mathbf{b}, \tag{7}$$

the magnetic curvature $\kappa = (\mathbf{b} \cdot \nabla)\mathbf{b}$ and the magnetic moment $\mu = V_\perp^2/(2B)$. In deriving Eq. (2), we have assumed, to simplify this presentation, that F_d is nearly isotropic and only relevant lowest-order terms are kept.

As noted by Hazeltine [1], the ω ordering in Eq. (2) represents a maximal ordering. That is, ω corresponds to both linear and nonlinear frequencies. In other words, if we decompose the fields into the background and fluctuating components, for example, $\mathbf{B} = \mathbf{B}_0 + \delta\mathbf{B}$, we correspondingly have $\dot{\mathbf{R}} = \dot{\mathbf{R}}_0 + \delta\dot{\mathbf{R}}$ and Eq. (1) implies that

$$|\omega| \equiv |\partial_t \ln |\delta\mathbf{B}|| \sim |\omega_{nl}| \equiv |\delta\dot{\mathbf{R}} \cdot \nabla \ln |\delta\mathbf{B}|| \sim \mathcal{O}(\epsilon)|\Omega|. \tag{8}$$

Equation (8), thus, establishes an upper bound on the fluctuation amplitude (e.g., $|\delta\mathbf{B}|/|\mathbf{B}|$). In the case of drift-kinetic equation, the background and fluctuating components could have comparable magnitudes. Consequently, as emphasized by Hazeltine [1], Eq. (2) does not involve the expansion in the distribution function or electromagnetic fields and thus facilitates, in principle, simulations on long time scales.

As to the nonlinear gyrokinetic equation, its main focus is on drift-type instabilities which, typically, peak at perpendicular wavelengths of the order of the thermal Larmor radius. Thus, the nonlinear gyrokinetic orderings are referred as [3, 4, 5, 6]

$$|\frac{\omega}{\Omega}| \sim |k_\parallel \rho| \sim |\frac{k_\parallel}{k_\perp}| \sim \mathcal{O}(\epsilon), \quad |k_\perp \rho| \sim \mathcal{O}(1), \tag{9}$$

and

$$|\frac{\delta f}{f}| \sim |\frac{\delta \mathbf{B}}{\mathbf{B}}| \sim \mathcal{O}(\epsilon). \tag{10}$$

It is worth emphasizing that the $|\delta\mathbf{B}|/|\mathbf{B}| \sim \mathcal{O}(\epsilon)$ ordering enters to ensure the validation of Eq. (8) even at $|k_\perp \rho| \sim \mathcal{O}(1)$; i.e.,

$$|\frac{\omega_{nl}}{\Omega}| \sim \frac{|\delta\dot{\mathbf{R}}|}{v_t} |k_\perp \rho| \sim |\frac{\delta\mathbf{B}}{\mathbf{B}}||k_\perp \rho| \sim \mathcal{O}(\epsilon). \tag{11}$$

Nonlinear gyrokinetic equation, thus, involves expanding physical quantities into background and finite but small fluctuating components.

For example, recently, nonlinear gyrokinetic equations have been derived in terms of electromagnetic fields [9, 10]. There, the distribution function is given by [10]

$$\langle f(\mathbf{x}, \mathbf{v}, t) \rangle = \langle \delta f_{pol} \rangle + \langle T_g^{-1} F_g \rangle, \tag{12}$$

where $\langle A\rangle$ represents gyrophase averaging of A, the polarization contribution $\langle\delta f_{pol}\rangle$ satisfies, by employing the Padé approximation,

$$(1-\frac{\rho^2\nabla_\perp^2}{2})\langle\delta f_{pol}\rangle=\frac{q}{m}\nabla_\perp\cdot\frac{\rho^2}{B_0}\frac{\partial F_0}{\partial\mu}\delta\mathbf{E}_\perp, \quad (13)$$

$F_0=F_0(\mathbf{X},\mu,U,t)$ is the background gyrocenter distribution function, $T_g=\exp(\rho\cdot\nabla_\perp)$, $\rho=\mathbf{b}_0\times\mathbf{v}/\Omega$ such that $\mathbf{x}=\mathbf{X}+\rho$, and $\mathbf{b}_0=\mathbf{B}_0/B_0$ with $\mathbf{B}_0$ being the background magnetic field. F_g, meanwhile, satisfies the following nonlinear gyrokinetic equation

$$(\partial_t+\dot{\mathbf{X}}\cdot\nabla+\dot{U}\partial_U)F_g(\mathbf{X},\mu,U,t)=0, \quad (14)$$

where the gyrocenter equations of motion are given by

$$\dot{\mathbf{X}}=U\mathbf{b}_0+\frac{1}{\Omega}\mathbf{b}_0\times(\mu\nabla B_0+U^2\mathbf{b}_0\cdot\nabla\mathbf{b}_0)+\langle\delta\mathbf{U}_g\rangle, \quad (15)$$

including the perturbed gyrocenter velocity

$$\langle\delta\mathbf{U}_g\rangle=[\frac{c}{B_0}\langle\delta\mathbf{E}_\perp\rangle-\frac{\mu}{\Omega}\nabla\langle\delta B_\parallel\rangle_*]\times\mathbf{b}_0+U\frac{\langle\delta\mathbf{B}_\perp\rangle}{B_0}, \quad (16)$$

and

$$\dot{U}=\frac{\mathbf{B}^*}{B_0}\cdot[\frac{q}{m}\langle\delta\mathbf{E}\rangle-\mu\nabla(B_0+\langle\delta B_\parallel\rangle_*)], \quad (17)$$

with the approximate modified magnetic field as

$$\mathbf{B}^*=\mathbf{B}_0+\langle\delta\mathbf{B}_\perp\rangle+\frac{UB_0}{\Omega}\mathbf{b}_0\times(\mathbf{b}_0\cdot\nabla)\mathbf{b}_0, \quad (18)$$

and $\langle A\rangle_*$ denoting the gyrophase average at the effective Larmor radius $\rho/\sqrt{2}$ [11]. Again, note that this set of nonlinear gyrokinetic equations corresponds to the expansions in terms of finite but small fluctuation amplitudes, i.e., $f=f_0+\delta f+\cdots$ and $\mathbf{B}=\mathbf{B}_0+\delta\mathbf{B}+\cdots$. That the background physical variables are evolving at slow nonlinear time scales implies that, over a long time scale, the background could deviate significantly from the initial state. This further indicates that an additional set of equations is needed to evolve the background variables [6, 12]; and, therefore, makes long-time simulations rather complex and difficult. It is then obviously desirable to derive a set of un-expanded nonlinear gyrokinetic equation, and this is the main goal and contribution of the present work.

2. Construction of Fully Nonlinear Gyrokinetic Model

To derive the un-expanded (or fully) nonlinear gyrokinetic equations, it is instructive to take the $|k_\perp^2\rho^2|\rightarrow 0^+$ limit of the expanded nonlinear gyrokinetic equations discussed above. We then have $|\langle\delta f_{pol}\rangle|=0^+$, $\langle A\rangle=\langle A\rangle_*=A$ and $\langle f(\mathbf{x},\mathbf{v},t)\rangle=F_g$. The gyrocenter phase space motion, correspondingly, reduces to

$$\dot{\mathbf{X}}=U[\mathbf{b}_0+\frac{\delta\mathbf{B}_\perp}{B_0}+\frac{1}{\Omega}\mathbf{b}_0\times(U\mathbf{b}_0\cdot\nabla\mathbf{b}_0)] +\frac{1}{\Omega}\mathbf{b}_0\times\mu\nabla(B_0+\delta B_\parallel)+\frac{c}{B_0}\delta\mathbf{E}_\perp\times\mathbf{b}_0, \quad (19)$$

and

$$\dot{U}=[\mathbf{b}_0+\frac{\delta\mathbf{B}_\perp}{B_0}+\frac{U}{\Omega}\mathbf{b}_0\times(\mathbf{b}_0\cdot\nabla)\mathbf{b}_0] \cdot[\frac{q}{m}\delta\mathbf{E}-\mu\nabla(B_0+\delta B_\parallel)]. \quad (20)$$

On the other hand, if we expand the drift-kinetic equations, Eqs. (2–7), by letting $\mathbf{E}=\delta\mathbf{E}$ and $\mathbf{B}=\mathbf{B}_0+\delta\mathbf{B}$, it can be readily shown that the lowest-order relevant terms recover Eqs. (19–20), and the resultant F_d recovers F_g in the $k_\perp^2\rho^2\rightarrow 0^+$ limit, as one would expect.

That the gyrokinetic F_g in the $k_\perp^2\rho^2\rightarrow 0^+$ limit recovers the drift-kinetic F_d in its expanded form suggests the interesting possibility of constructing nonlinear electromagnetic gyrokinetic equations without expanding the physical quantities into background and finite but small-amplitude fluctuating components. More specifically, examining the unexpanded drift-kinetic equations, Eqs. (2–7), and the expanded nonlinear gyrokinetic equations, Eqs. (14–18), readily suggests the following unexpanded (fully) nonlinear electromagnetic gyrokinetic equations:

$$\langle F\rangle=\langle F_{pol}\rangle+\langle T_g^{-1}F_g\rangle, \quad (21)$$

where the polarization contribution

$$F_{pol}=\frac{q}{m}[\phi-T_g^{-1}\langle\phi\rangle]\frac{\partial}{B\partial\mu}T_g^{-1}F_g \quad (22)$$

can be conveniently expressed, using the Padé approximation and $\mathbf{E}_\perp=-\nabla_\perp\phi$, as the reduced form more suitable for numerical implementation

$$(1-\frac{\rho^2\nabla_\perp^2}{2})\langle F_{pol}\rangle=\frac{q}{m}\nabla_\perp\cdot[\frac{\rho^2}{B}\frac{\partial}{\partial\mu}\langle T_g^{-1}F_g\rangle\mathbf{E}_\perp]. \quad (23)$$

Thus, the Padé approximation has the scope of rendering an integral equation into an approximated partial differential equation (PDE). Being this procedure based on an approximation, other choices of Eq. (23) are possible if the intended application is focused on a specific physics problem. Meanwhile, the gyrocenter response is determined by

$$(\partial_t+\dot{\mathbf{X}}\cdot\nabla+\dot{U}\partial_U)F_g(\mathbf{X},\mu,U,t)=0, \quad (24)$$

with the gyrocenter phase space motion

$$\dot{\mathbf{X}}=U\frac{\mathbf{B}_g^*}{B_{g\parallel}^*}+\mathbf{U}_B+\mathbf{U}_E, \quad (25)$$

$$\mathbf{U}_B=\frac{\mu B}{\Omega B_{g\parallel}^*}\mathbf{b}_g\times\nabla\langle B_g\rangle_*, \quad (26)$$

$$\mathbf{U}_E=\frac{c\langle\mathbf{E}_\perp\rangle\times\mathbf{b}_g}{B_{g\parallel}^*} \quad (27)$$

and

$$\dot{U}=\frac{\mathbf{B}_g^*}{B_{g\parallel}^*}\cdot[\frac{q}{m}\langle\mathbf{E}\rangle-\mu\nabla\langle B_g\rangle_*]. \quad (28)$$

In the unexpanded gyrokinetic formulation, the modified magnetic field has the form

$$\mathbf{B}_g^* = \mathbf{B}_g + \frac{UB}{\Omega}\nabla\times\mathbf{b}_g, \qquad (29)$$

where $\mathbf{B}_g = \langle\mathbf{B}\rangle \equiv B_g\mathbf{b}_g$ and $B_{g\|}^* = \mathbf{B}_g\cdot\mathbf{b}_g$. We emphasize, again, that the above fully nonlinear electromagnetic gyrokinetic equations, as constructed, contain the complete finite-Larmor-radius effects and are un-expanded; and, therefore, remain valid, in principle, for fluctuating amplitudes comparable to the background ones. Furthermore, as noted earlier, without expanding variables into background and fluctuating components, no separate background and fluctuation equations are needed; which could significantly expedite simulations over long-time scales. In addition, since these equations reduce to the proper drift-kinetic equations in the long-wavelength limit, they are applicable to simulations covering the entire microscopic to macroscopic spatial scales.

It is certainly desirable to generalize these equations to include terms due to velocity-space anisotropy. Taking the expanded nonlinear gyrokinetic equations [6] and follow the approaches of [10] and above, we can readily construct that, with anisotropy, Eq. (21) is modified to

$$\langle F\rangle = \langle F_{pol}\rangle + \langle T_g^{-1}F_g\rangle + \langle F_{mm}\rangle + \langle F_{fh}\rangle, \qquad (30)$$

where

$$\langle F_{mm}\rangle = -\langle T_g^{-1}[\mu\langle B_g\rangle_*(\frac{\partial}{B_g\partial\mu} - \frac{\partial}{U\partial U})F_g]\rangle,$$

and, with the Padé approximation,

$$(1-\frac{\rho^2\nabla_\perp^2}{2})\langle F_{fh}\rangle$$
$$= -\frac{q}{m}\mathbf{b}\cdot\nabla_\perp\times[\frac{\rho^2}{2}(\frac{\partial}{B\partial\mu} - \frac{\partial}{U\partial U})\langle T_g^{-1}F_g\rangle\frac{U\mathbf{B}}{c}].$$

$\langle F_{pol}\rangle$ and F_g, meanwhile, remain the same as given by Eqs. (23) and (24). Anisotropy, thus, contributes to, respectively, the additional mirror-mode $\langle F_{mm}\rangle$, and fire-hose $\langle F_{fh}\rangle$ terms, as we shall anticipate. Also, from the above derivation, we note that, in the drift-kinetic equations [1], the anisotropy terms $\propto$ $[\partial/(B\partial\mu) - \partial/(U\partial U)]F_d$ are, typically, negligible.

3. Conclusions

In conclusion, we have constructed a set of fully (un-expanded) nonlinear electromagnetic gyrokinetic equations. These equations asymptotically agree with the drift-kinetic equation in the long-wavelength limit, and the usual nonlinear gyrokinetic equation in the expanded short-wavelength limit. They are, thus, applicable to simulations covering the wide range of macroscopic to microscopic scales. In the long wavelength limit, for example, they recover various forms of collisionless MHD equations as insightfully discussed in Ref [2]. In addition, since these equations are constructed without expanding into background and small-amplitude fluctuating component, we submit that they are suitable for long-time stimulations where the background plasma state could evolve sufficiently away from the initial state. Finally, we note that, as these equations are constructed *ad hoc* via asymptotic matching with the various limiting cases, it will be interesting to derive/improve these fully nonlinear electromagnetic gyrokinetic equations based on fundamental analytical approaches. A more thorough derivation should also include any additional higher order terms to ensure proper conservation theorems, which can be crucial in long-time simulations. Such an analysis should be pursued in the future.

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