# A Quantum Kinetic Solver: Theory, Structure and Simulation

Tian-Xing Hu[a], Jiong-Hang Liang[a], Dong Wu[b,*], Zheng-Mao Sheng[a,**]

[a] *Institute for Fusion Theory and Simulation, Department of Physics, Zhejiang University, Hangzhou 310027, China.*

[b] *Key Laboratory for Laser Plasmas and School of Physics and Astronomy, Collaborative Innovation Center of IFSA, Shanghai Jiao Tong University, Shanghai 200240, China.*

## Abstract

High-Energy-Density-Plasma differs from classical plasma in the follow two respects: (1) The statistical equilibrium become Fermi-Dirac-like instead of Maxwellian; (2) The quantum nature of single electron, i.e., the wave effect, has to be considered. The Wigner equation is the quantum version of Vlasov equation, the former is more general than the latter. However the solution of Wigner equation is non-trivial. Here we have adopted a hybrid splitting scheme in a Eulerian grid, where the $x$ and $v$ direction of the phase space are advanced by different methods. The hybrid scheme shows significant improvements when compared with the typical splitting scheme, especially when the non-linear interactions become serious. The linear results with temperature effect is tested, we found that the extra unstable region of two-stream instability is suppress by kinetic effect except when the quantum parameters are in a certain range. The quantum nonlinear Landau damping is also presented.

*Keywords:*
HEDP; Wigner Equation;

## PROGRAM SUMMARY

***Program Title:*** *QUAKINS*

***Programming language:*** *C++*

***Nature of problem:*** *Wigner equation describes the behavior of quantum collisionless plasmas just like the Vlasov equation describes the classical plasmas. Quakins provide a general numerical solution of the Wigner equation efficiently, where a long-time nonlinear simulation is also supported.*

* *dwu@sjtu.edu.cn*
** *zmsheng@zju.edu.cn*

***Solution method:*** *Solving the Vlasov/Wigner equation via splitting method.*
***Restrictions:*** *Collisional and electromagnetic effects are not included.*

## 1. Introduction

In a kinetic theory, state of plasmas is indicated by a distribution function $f(\mathbf{x}, \mathbf{v}, t)$, which evolves with time according to the famous Vlasov or Boltzmann equation. Solving the Vlasov/Boltzmann equation is one of the most important tasks for a plasma physicist. However, a nonlinear Boltzmann equation is almost impossible to solve analytically, numerical means are often needed. As a 6-dimension phase space fluid (PSF), the numerical methods can be categorized into two types analog to computational fluid dynamics: the Lagrangian and Euler method. In Lagrangian perspective, the PSF is marked by a huge amount of space point travelling with corresponding velocity. Those points are called "markers" in magnetic fusion literature and "marco-particles" or "clouds of particle" in the field of high density plasma. This kind of method is known as the Particle-in-Cell (PIC) method. A major defect of PIC is the inevitable noise caused by the Monte-Carlo process. On the contrary, Euler methods are noiseless, but it is computationally unaffordable when the dimension of the simulated space is higher than two.

In the field of High-Energy-Density-Plasmas (HEDP), traditional kinetic methods fail for the quantum effects become dominant. Many-body quantum mechanics is one of the most popular approaches to deal with quantum plasmas [1], which usually requires directly solving the many-body wave-function. However, it could not produce dynamical results [2]. An alternative approach is the quantum kinetic theory (QKT), which is essentially just a different form of many-body Schrödinger equation. The governing equation of QKT is the quantum Vlasov equation, also known as the Wigner equation [3]. Numerical methods solving classical phase space problem, namely, solving the Vlasov or Boltzmann equation have been vastly investigated in the past decades. Hence the quantum effect can be introduced as a correction to approximately describe a mesoscopic system. Furthermore, since the phase-space-based method is a real time-dependent Hartree mean-field theory, it is able to deal with dynamic correlations, i.e., collisions, among distribution functions [2]. The collisional process are easily introduced into the phase-space framework by simply add a collision term at the right-hand-side of kinetic equation, which can be solved by directly solving [4], or Monte-Carlo relaxion method, depending on the form of collision term.

To solve the nonlinear Wigner equation, Fourier spectrum method (FSM) [3] is often needed. However, in the nonlinear stage, the phase space calculated by FSM become unphysically chaotic. We thus adopt a combined method in this paper to remove this unphysical issues.

This paper is organized as follows. In section 2, the basic theory and numerical algorithms are introduced. We adopted a hybrid splitting scheme in the `quakins` code to solve the Wigner equation. The code structure is briefly introduced in section 3. Results of `quakins` are presented in section 4, including both the classical and the quantum result of Landau damping and two-stream instability. Some riveting nonlinear phenomena are also presented.

## 2. Basic Theory and Methods

### 2.1. Quantum Kinetic Theory

In simple terms, the Wigner equation [**?** ]

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}\right) f(\mathbf{x},\mathbf{v},t) = \frac{1}{i\mathcal{Q}}\int \mathrm{d}\boldsymbol{\xi}\int \frac{\mathrm{d}\mathbf{v}'}{(2\pi\mathcal{Q})^3} \times e^{i(\mathbf{v}'-\mathbf{v})\cdot\boldsymbol{\xi}/\mathcal{Q}}\left[\phi\left(\mathbf{x}+\frac{\boldsymbol{\xi}}{2}\right) - \phi\left(\mathbf{x}-\frac{\boldsymbol{\xi}}{2}\right)\right] f(\mathbf{x},\mathbf{v}',t) \tag{1}$$

is just the quantum version of the electrostatic Vlasov equation:

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla} - \boldsymbol{\nabla}\phi\cdot\frac{\partial}{\partial \mathbf{v}}\right) f(\mathbf{x},\mathbf{v},t) = 0. \tag{2}$$

One can see that they are identical when $\phi(\mathbf{x})$ varying slowly in space. Here,

$$\mathcal{Q} = \frac{\hbar\omega_{\mathrm{p}}}{2k_{\mathrm{B}}T} \tag{3}$$

is the normalized Planck's constant, which, in quantum kinetic theory, measures the importance of quantum effect. Noticing that there is a intrinsic difference between Eq. (1) and Eq. (2): the definition of $f$ in Eq. (1) is not a distribution function, i.e., a probability distribution, but

$$f(\mathbf{x},\mathbf{v},t) = \int \mathrm{d}\boldsymbol{\xi} e^{-im\mathbf{v}\cdot\boldsymbol{\xi}/\hbar}\left\langle \Psi^\dagger\left(\mathbf{x}-\frac{\boldsymbol{\xi}}{2}\right)\Psi\left(\mathbf{x}+\frac{\boldsymbol{\xi}}{2}\right)\right\rangle, \tag{4}$$

where $\Psi$ is the quantum field operator, and $\langle\cdots\rangle$ stands for ensemble average. Eq. (4) is known as *the Wigner quasi-distribution function* [5]. The Wigner function must not be interpreted as a probability distribution, because it can have negative values. In fact, this is the most significant difference between the quantum kinetic theory and traditional kinetic theory. Negative values imply the presence of quantum coherence in high-density plasmas.

### *2.2. Numerical Methods*

The most common method for solving a Vlasov equation in Euler grids is to split a single calculating step into two parts [6], which are in $x$ and $v$ direction respectively. The flux balance method (FBM) [7] is commonly used when solving the Vlasov, while the Wigner equation can only be solved by Fourier spectrum method (FSM) [3] due to the cumbersome phase space integral. We adopt a novel scheme that combines these two methods, and the advantages of which will be presented in the following sections. Along with the Poisson equation solving process, a completed step of the main loop of the `quakins` code is in a leap-frog-like form.

#### *2.2.1. Flux Balance Method*

To solve a continuity equation

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial x}(vf) = 0, \tag{5}$$

one may discretize $f(x)$ into $N$ uniform distributed grid point, and let

$$f_i = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} f(x)\mathrm{d}x. \tag{6}$$

In light of the nice property of Eq. (5), the integration in Eq. (6) is conserved along the characteristic curves. Hence,

$$\int_{x_{i-1/2}}^{x_{i+1/2}} f(x,t^b)\mathrm{d}x = \int_{\mathcal{X}_{b\to a}(x_{i-1/2})}^{\mathcal{X}_{b\to a}(x_{i+1/2})} f(x,t^a)\mathrm{d}x, \tag{7}$$

where $\mathcal{X}_{b\to a}$ maps a space location at time $t^b$ to its whereabout at time $t^a$ along characteristic curve [7]. In this case $\mathcal{X}_{b\to a}(x) = x - v(t^b - t^a)$. Defining

$$\Phi_k^{ba} = \int_{\mathcal{X}_{b\to a}(x_k)}^{x_k} f(x,t^a)\mathrm{d}x, \tag{8}$$

which is the total volume that flow through $x_k$ point from $t^a$ to $t^b$, then we have

$$f_i(t^b)\Delta x = f_i(t^a)\Delta x + \Phi_{i-1/2}^{ba} - \Phi_{i+1/2}^{ba}, \tag{9}$$

which is a perfect discretization of the continuity equation (5). Noticing that Eq. (9) indicate that the change in total volume in a single cell from $t^a$ to $t^b$ is equal to the volume flow in from left boundary($x_{i-1/2}$) minus the volume flow out from right boundary($x_{i+1/2}$). The flux integral of all the boundaries can then be evaluated independently:

$$\Phi_{i+\frac{1}{2}}(t^n) = \int_{x\in\alpha_{i+1/2}} f(t^n,x)\mathrm{d}x + \begin{cases} \sum_{a=I+1}^{i} f_a^n \Delta x, & \text{while } v>0 \\ -\sum_{a=i+1}^{I-1} f_a^n \Delta x, & \text{while } v<0 \end{cases}. \tag{10}$$

In `quakins` code, the first integration term in Eq. 10 can be calculated by interpolation of any order of accuracy.

*2.2.2. Fourier Spectrum Method*

The flux balance method can easily handle a Vlasov equation but not the Wigner equation because of the cumbersome phase space integration term. However, it turn out that this term become more clear in Fourier space [3]:

$$\frac{\partial}{\partial t} f_\lambda(x,t) = \frac{e}{i\hbar}\left[\phi\left(x+\frac{\hbar\lambda}{2m}\right) - \phi\left(x-\frac{\hbar\lambda}{2m}\right)\right] f_\lambda(x,t). \tag{11}$$

Here, $\lambda$ is the Fourier conjugate of velocity $v$. The solution of Eq. (11) is

$$f_\lambda(t) = f_\lambda(t-\Delta t)\exp\left\{\frac{e}{i\hbar}\left[\phi\left(x+\frac{\hbar\lambda}{2m}\right) - \phi\left(x-\frac{\hbar\lambda}{2m}\right)\right]\Delta t\right\}. \tag{12}$$

Similarly, the $x$ direction advance equation is in $k$-Fourier space:

$$f_k(t) = f_k(t-\Delta t)e^{ikv\Delta t}. \tag{13}$$

This implies that one can solve the Wigner function by means of Fourier spectrum method.

*2.2.3. hybrid splitting Method*

In `quakins`, we adopt a hybrid splitting method, in which, the $x$-direction is advance by FBM method while the $v$-direction the FSM method. Then a full step of the main loop is

$$\begin{aligned} f^*(x,v) &= f\left(x - v\frac{\Delta t}{2}, v, t-\Delta t\right), \\ \mathfrak{F}\left[f^{**}\right](\lambda) &= \mathfrak{F}\left[f^*\right](\lambda) e^{i\mathcal{E}(x)\lambda\Delta t}, \\ f(x,v,t) &= f^{**}\left(x - v\frac{\Delta t}{2}, v\right), \end{aligned} \tag{14}$$

where $\mathfrak{F}$ stands for a Fourier transformation, and

$$\mathcal{E}(x) = \begin{cases} -\partial\phi/\partial x, & \text{while } \mathcal{Q} = 0, \\ \left[\phi\left(x + \mathcal{Q}\lambda/2\right) - \phi\left(x - \mathcal{Q}\lambda/2\right)\right]/\mathcal{Q}, & \text{otherwise.} \end{cases} \tag{15}$$

From Fig. 1 one can see that, the total energy, which equals to the sum of the electrostatic wave(ESW) energy ($\propto |E|^2$) and particle energy ($\propto \int v^2 f \mathrm{d}v$), calculated by the pure FSM and hybrid method are both conserved. And, the linear stage of the ESW of these two Methods perfectly coincide, but disagreed in the nonlinear stage. Noticing that the two both predicted a sudden collapse of ESW in the nonlinear saturation stage, but the collapse time are discrepant. This collapse is actually a nonlinear side-band instability caused by mode-mode coupling [8], which indicates the broken of a Bernstein-Greene-Kruskal(BGK) equilibrium [9]. To see which mean is more reliable, we shall take a look at the phase space. In Fig. 2, where snap-shot of a typical BGK holes calculated by the two means respectively are presented, one can see that the FSM phase space is quite noisy, while the hybrid is fairly smooth.

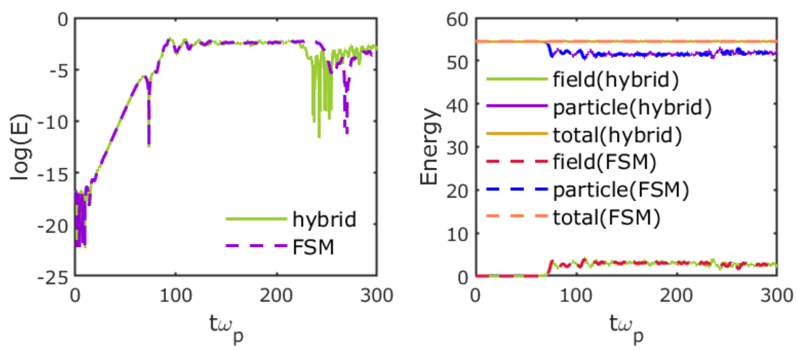


Figure 1: Electric field at a random position(left) and the energy of plasma and wave (right) of a typical two-stream instability.

### *2.2.4. Hypercollision Operator*

As the simulation goes on, the ballistic term($\propto e^{ikvt}$) create a increasingly high wave-number of velocity space. In real world, this term will automatically fade away in light of phase mixing. However, in discrete phase space, the velocity space integral is replaced by a finite summation. As a result, the Fourier integral degrades to sum of Fourier series, hence a physical quantity become periodic in time. This is the so-called recurrence problem [6].

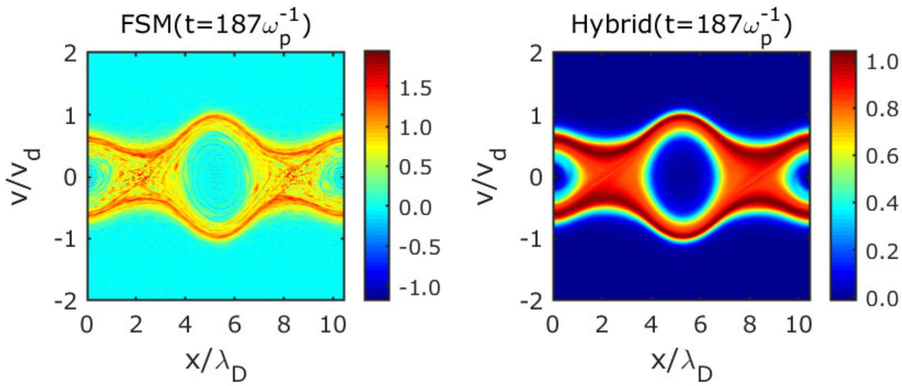


Figure 2: Snap-shot of two-stream instability at nonlinear phase simulated by pure fft method and hybrid method respectively, where $n_x = 200, n_v = 201, k = 0.6$.

We introduce a hypercollision operator [10]

$$C(f) = \nu_h \frac{\partial^4 f}{\partial v^4} \tag{16}$$

to cope with the recurrence problem.

## 3. Code Structure

The `quakins (QUAntum KINetic Solver)` code is developed in C++ language, and used for solving the 1d nonlinear Wigner/Poisson system in Euler grids with periodic boundary condition. We adopt the `MPI` parallelization scheme in order to do quick parameter scanning. The algorithm repository of `quakins` is designed to be easily expandable. The Poisson's equation solver and the free stream solver are two pure virtual interfaces, which allow one to enrich the algorithm repository via simply adding a subclass of the pure virtual base class (see Fig. 3).

## 4. Simulation Results

### *4.1. Classical Results($\mathcal{Q} = 0$)*

Landau damping is one of the most common kinetic phenomenon in plasma physics. Here we carry out the analytical continuation of the plasma dispersion function [11]

$$Z(\zeta) = i\sqrt{\pi}e^{-\zeta^2}\left[1 + \mathrm{erf}(i\zeta)\right] \tag{17}$$

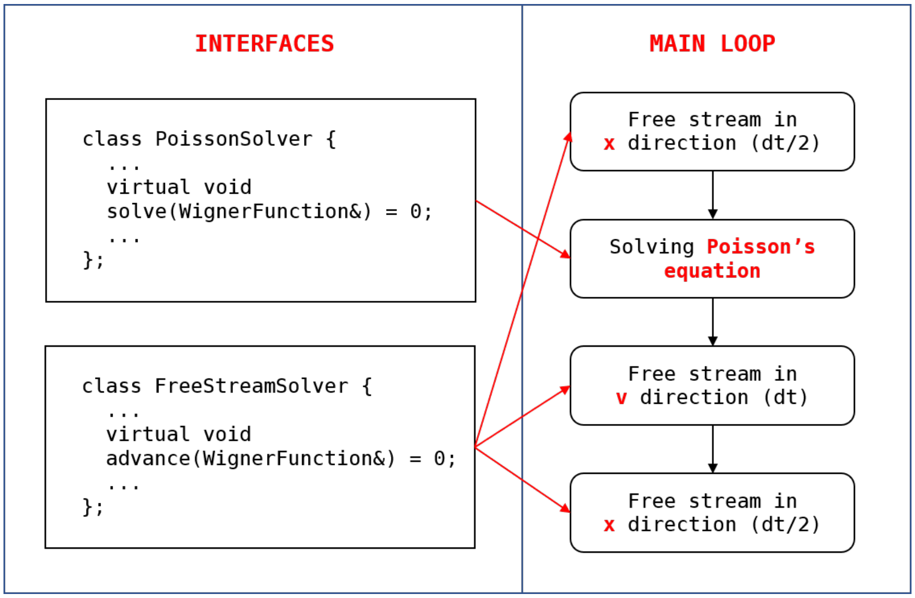


Figure 3: Quakins code as an API.

to calculate the exact solution of classical Landau damping (solid lines in Fig. 4(a)). The initial perturbation of the numerical result is of the form

$$f(x, v, 0) = f_0(v)\left[1 + A\cos(k_0 x)\right]. \tag{18}$$

To minimize unwanted nonlinear effects, we set $A = 10^{-3}$ for Landau damping. One can see from Fig. 4(a) that the numerical results perfectly coincide with the exact solution. Fig. 4(a) also shows the analytic solution obtained from Landau integral (the dash-lines), which is only valid in a very limited region ($k\lambda_{\mathrm{D}} \ll 1$). Fig. 4(b) present the result of a typical two-stream instability. To the linear limit, the two-stream instability is not a kinetic effect, and the solid-lines in Fig. 4(b) is the solution to the reactive-type dispersion relation:

$$1 - \frac{\omega_{\mathrm{p}}^2}{(\omega - k v_{\mathrm{d}}/2)} - \frac{\omega_{\mathrm{p}}^2}{(\omega + k v_{\mathrm{d}}/2)} = 0. \tag{19}$$

In simulation, we initialized two nearly cold beams ($v_{\mathrm{th}}/v_{\mathrm{d}} = 0.05$) with opposite velocity ($v/v_{\mathrm{d}} = \pm 0.5$), where the kinetic effects should be negligible. When the initial perturbation $A = 10^{-6}$, one can see that the simulation results do coincide with the analytical solution.

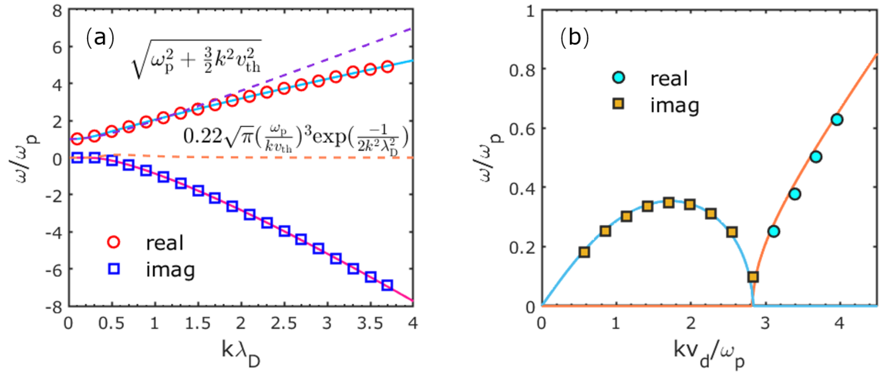


Figure 4: Linear growth rate of (a)Landau damping and (b)two-stream instability, where the solid lines are calculated by analytical equation, and the dash-lines are from Landau's approach.

### *4.2. Quantum Results($\mathcal{Q} > 0$)*

#### *4.2.1. Landau Damping*

When it comes to wave-particle interaction in kinetic theory, one needs to only consider the velocity component that parallel to the wave vector. Hence, after the integration over other two dimension, one obtain the one-dimensional Fermi-Dirac distribution

$$f(v) = \frac{3}{4}\frac{n_0}{v_{\rm F}}\Theta \ln\left\{1+\exp\left[\frac{1}{\Theta}\left(1-\frac{v^2}{v_{\rm F}^2}\right)\right]\right\}, \tag{20}$$

where $\Theta = k_{\rm B}T/\epsilon_{\rm F}$ is the degeneracy of a Fermi-Dirac system. Noticing that the degeneracy is related to the normalized Planck's constant $\mathcal{Q}$ by

$$\Theta = 1.6813 \times 10^4 n^{-\frac{1}{6}}\mathcal{Q}^{-1}. \tag{21}$$

Hence, for a fixed electron density $n$, the shape of Fermi-Dirac distribution function varies with $\mathcal{Q}$.

The dispersion relation presented in Fig. 5 is calculated under Maxwellian distribution, thus the degenerate effect is ignored. The dash-lines in the left panel of Fig. 5 stand for quantum fluid approximation [12], i.e.,

$$\omega^2 = \omega_{\rm p}^2 + k\langle v\rangle^2 + \frac{\hbar^2 k^4}{4m_e^2}, \tag{22}$$

which, just like in the classical case, overestimated the real frequency. In Fig. 6, the effect of Fermi-Dirac statistics are included. We calculated the dispersion relation with $\Theta = 0.2$, 0.5 and 2 respectively. As can be seen, the shape of the distribution function has a significant effect on the dispersion relation.

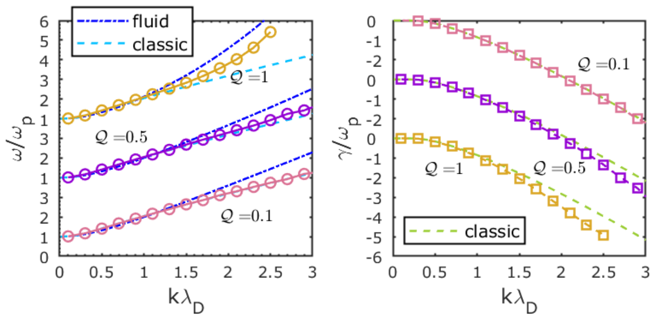


Figure 5: quantum Landau Damping with varying $\mathcal{Q}$.

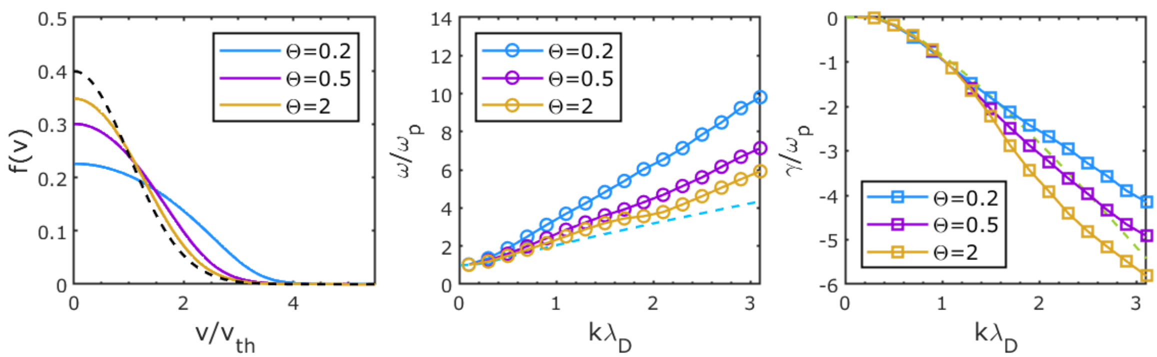


Figure 6: Quantum Landau Damping with varying *Theta*. The first panel is the shape of 1d Fermi-Dirac distribution, where the dashed-line represents the Maxwellian distribution.

#### *4.2.2. Two-stream Instability*

In quantum degenerate plasma, two-stream instability behave differently because of the Fermi pressure and the quantum wave effect [13, 14, 15, 16]. When the relative drift velocity of the two streams $v_{\mathrm{d}}$ is much larger than the Fermi velocity $v_{\mathrm{F}}$ or the thermal velocity $v_{\mathrm{th}}$ in non-degenerate plasma, it is reasonable to replace the distribution function by two counter-streaming $\delta$-functions with velocity difference $v_{\mathrm{d}}$. Since the kinetic effect is ignored, a reactive-type dispersion relation is obtained:

$$1 - \frac{\omega_{\mathrm{p1}}^2}{(\omega - kv_1)^2 - \omega_k^2} - \frac{\omega_{\mathrm{p2}}^2}{(\omega - kv_2)^2 - \omega_k^2} = 0, \tag{23}$$

where $\omega_k = \hbar k^2/2m_e$ is the quantum shift caused by diffraction and refraction of electrons. Let $v_\mathrm{d} = v_1 - v_2$, $\omega_1^2 = \omega_2^2 = \omega_\mathrm{p}^2/2$, and $v_1 + v_2 = 0$, we have

$$\omega^2 = \frac{\omega_\mathrm{p}^2}{2}\left(1 + 2\tilde{k}^2 + 2H^2\tilde{k}^4 \pm \sqrt{1 + 8\tilde{k}^2 + 16H^2\tilde{k}^6}\right), \tag{24}$$

where $H = 2\hbar\omega_\mathrm{p}/mv_\mathrm{d}^2$, and $\tilde{k} = kv_\mathrm{d}/2\omega_\mathrm{p}$ is the normalized wave number. The plus sign gives a trivial solution since the frequency is always real. The imaginary part of roots with minus sign are plotted in Fig. 7, from which one can see that as $H$ increases, there is an additional unstable bubble emerges at high-$k$ and then merges with the original bubble as $H$ further increases. When $H = 0$, the bubble is located at infinite-$k$. This additional unstable bubble would be better understood if the electromagnetic effect were considered [13]. Generally, an instability with such high value of wave number often suffers very strong Landau damping and being difficult to really grow up.

We thus conducted a simulation at $H \simeq 0.5$, while the two instability bubble are at the edge of merging. The temperature of the two beams are both $0.05v_\mathrm{d}$, which corresponds to a low temperature system. The results are presented in right panel of Fig. 7. When $H = 0.48$, the unstable mode in the outer bubble is suppress by Landau damping albeit with such low temperature. When $H = 0.52$, where the outer bubble is attached to the main unstable bubble, and a larger unstable region is formed. The numerical result coincide with the theoretical result up to $k = 4\omega_\mathrm{p}/v_\mathrm{d}$ while the remaining part is again suppressed by Landau damping. Be that as it may, the unstable region of QTS is still almost doubled when compared to the CTS (see Fig. 4).

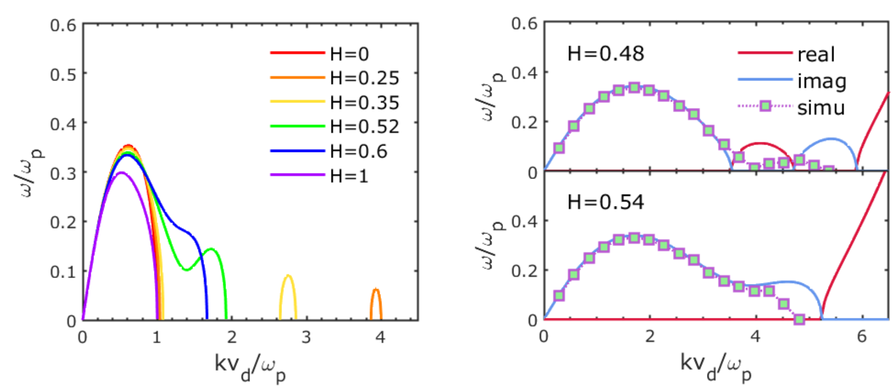


Figure 7: quantum two-stream instability.

### *4.3. Nonlinear Effects*

In classical plasmas, a strong electrostatic perturbation can not be completely removed by Landau damping because of nonlinear effect, i.e., the particle trapping process. However, in a high-density-plasma, quantum tunneling effect prevent particles from being trapped by an electrostatic trough when the relation [17]

$$\frac{\hbar k}{m} \gtrsim \frac{\omega_{\mathrm{p}}}{k}\sqrt{A} \tag{25}$$

is satisfied. More specifically, as is pointed out by the author of Ref. [18], there exists two time scales that determine the nonlinear behavior: the bounce period $t_{\mathrm{B}} = 2\pi/\omega_{\mathrm{B}}$, where $\omega_{\mathrm{B}}$ is the bounce frequency of electrons, and the quantum time scale $t_{\mathrm{Q}} = 2m/\hbar k^2$. When $t_{\mathrm{Q}} \ll t_{\mathrm{B}}$, the nonlinear trapping is suppressed, and when $t_{\mathrm{Q}} \gg t_{\mathrm{B}}$, the quantum result reduce to classical.

We then set $A = 0.06$ to initial a Landau damping with $\mathcal{Q} = 0$ and 1 respectively. One can see from Fig. 8 that, when $k\lambda_{\mathrm{D}} \simeq 0.6$, $t_{\mathrm{Q}} \ll t_{\mathrm{B}}$ is satisfied, the nonlinear trapping does be suppressed with no residue left. However, when $k\lambda_{\mathrm{D}} \simeq 0.4$, where $t_{\mathrm{Q}}$ is just slightly less than $t_{\mathrm{B}}$, the quantum result resembles the classical. The nonlinear effect is still evident. If we take a look at the snap-shot of phase space (additional panel in Fig. 8) at $t\omega_{\mathrm{p}} = 52$, where the trapped electrons are in their first bounce period, we find that the resonant island of the quantum plasma, where the negative value indicates the quantum recoil, behaves very differently than its classical counterpart, albert the evolution of electric field nearly coincides. This implies that the quantum recoil phenomenon may not be as significant as the theoretical analysis has predicted.

## 5. Summary

In this paper, we adopt a hybrid splitting method to solve the Wigner equation, the accuracy of which is benchmarked by analytical linear theory. We have re-investigated the two famous phenomena in plasma physics: Landau damping and two-stream instability. The result shows that the Landau damping is much stronger in quantum plasmas than in classical plasma for two reasons: the quantum wave effect and the Fermi pressure. As to quantum two-stream instability, we find that the extra short-wave-length instability resulted by quantum recoil effect is suppress by strong Landau damping even with nearly zero temperature. And only exist when it is closely located at the original unstable region, which happens when $H \simeq 0.5$. In the nonlinear region, we conclude that the quantum recoil effect may not be as that important as the prediction of theoretical analysis.

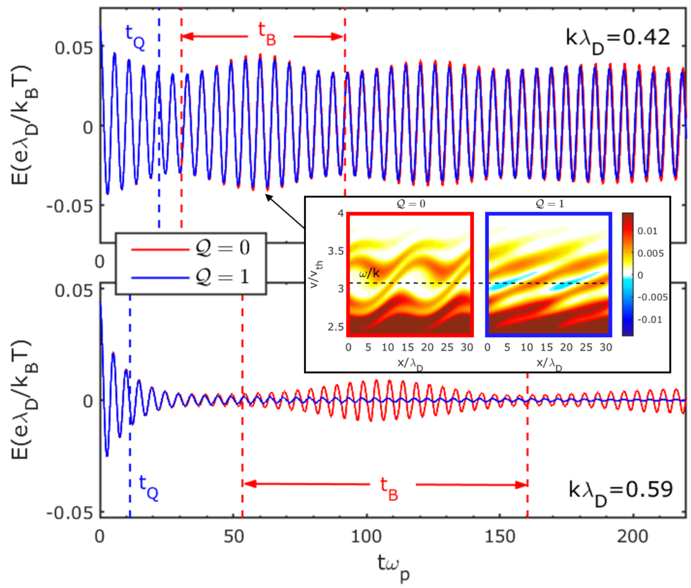


Figure 8: Nonlinear Landau Damping.

## 6. Acknowledgements

This work was supported by National Natural Science Foundation of China 11875235 and 61627901.

## References


[1] T. Dornheim, S. Groth, M. Bonitz, The uniform electron gas at warm dense matter conditions, Physics Reports 744 (2018) 1–86.

[2] G. Manfredi, P.-A. Hervieux, J. Hurst, Phase-space modeling of solid-state plasmas, Reviews of Modern Plasma Physics 3 (1) (2019) 1–55.

[3] N.-D. Suh, M. R. Feix, P. Bertrand, Numerical simulation of the quantum liouville-poisson system, Journal of Computational Physics 94 (2) (1991) 403–418.

[4] J. Daligault, On the quantum landau collision operator and electron collisions in dense plasmas, Physics of Plasmas 23 (3) (2016) 032706.

[5] G. Manfredi, P.-A. Hervieux, J. Hurst, Phase-space modeling of solid-state plasmas, Reviews of Modern Plasma Physics 3 (1) (2019) 13.

[6] C.-Z. Cheng, G. Knorr, The integration of the vlasov equation in configuration space, Journal of Computational Physics 22 (3) (1976) 330–351.

[7] F. Filbet, E. Sonnendrücker, P. Bertrand, Conservative numerical schemes for the vlasov equation, Journal of Computational Physics 172 (1) (2001) 166–187.

[8] V. E. Zakharov, Collapse of langmuir waves, Soviet Physics JETP 35 (5) (1972) 908–914.

[9] I. B. Bernstein, J. M. Greene, M. D. Kruskal, Exact nonlinear plasma oscillations, Physical Review 108 (3) (1957) 546.

[10] T. Dannert, F. Jenko, Vlasov simulation of kinetic shear alfvén waves, Computer physics communications 163 (2) (2004) 67–78.

[11] H.-S. Xie, Generalized plasma dispersion function: One-solve-all treatment, visualizations, and application to landau damping, Physics of Plasmas 20 (9) (2013) 092125.

[12] G. Manfredi, F. Haas, Self-consistent fluid model for a quantum electron gas, Physical Review B 64 (7) (2001) 075316.

[13] A. Bret, F. Haas, Connection between the two branches of the quantum two-stream instability across the k space, Physics of Plasmas 17 (5) (2010) 052101.

[14] S. Son, Two-stream instabilities in degenerate quantum plasmas, Physics Letters A 378 (34) (2014) 2505–2508.

[15] F. Haas, B. Eliasson, A new two-stream instability mode in magnetized quantum plasma, Physica Scripta 90 (8) (2015) 088005.

[16] J.-H. Liang, T.-X. Hu, D. Wu, Z.-M. Sheng, Kinetic study of quantum two-stream instability by wigner approach, Physical Review E 103 (3) (2021) 033207.

[17] G. Manfredi, How to model quantum plasmas, Fields Institute Communications 46 (2005) 263–287.

[18] J. Daligault, Landau damping and the onset of particle trapping in quantum plasmas, Physics of Plasmas 21 (4) (2014) 040701.