Self-Consistent Kinetic Theory With Nonlinear Wave-Particle Resonances

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Abstract. We have developed, based on the oscillating-center transformation, a general theoretical approach for self-consistent plasma dynamics including, explicitly, effects of nonlinear (higher-order) wave-particle resonances. A specific example is then given for low-frequency responses of trapped particles in axisymmetric tokamaks. Possible applications to transport as well as nonlinear wave growth/damping are also briefly discussed.

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1. Introduction

It is well-known that wave-particle resonances play crucial roles in the dynamics of plasmas. That is, wave-particle resonances can lead to efficient exchanges of energy and momenta between waves and particles [1, 2, 3, 4, 5, 6] (see, *e.g.*, the recent review [7]). As a consequence, wave-particle resonances can lead to, for example, wave growth/damping, and heating/acceleration as well as transports of charged particles. To quantitatively analyze these effects, one needs to employ a self-consistent kinetic theory including terms due to wave-particle resonances. Up to now, corresponding theoretical treatments have been limited to those associated with linear (primary) wave-particle resonances, including possible nonlinear modifications; such as wave trapping or resonance broadening. On the other hand, in the presence of finite-amplitude electromagnetic fluctuations, studies on the single-particle dynamics have revealed that nonlinear (higher-order) wave-particle resonances [8, 9] could also play important dynamic roles in the heating/acceleration [10, 11] as well as transports of charged particles [12, 13, 14]. To our knowledge, there is, however, no corresponding self-consistent kinetic treatment; which constitutes the primary motivation of the present work.

In this work, we first propose an oscillating-center transformation approach and analyze a general theoretical model, demonstrating explicitly the presence of nonlinear wave-particle resonances. This analysis is presented in Sec. 2. In Sec. 3, a specific example pertinent to self-consistent resonances of magnetically trapped particles due to low-frequency fluctuations in an axisymmetric tokamak is presented. Section 4 contains a summary and discussion of possible applications to transport and wave growth/damping via nonlinear wave-particle resonances. Appendix A, finally, provides a detailed analysis on nonlinear wave-particle resonances in an axisymmetric tokamak.

2. General theoretical approach

Let $f(\boldsymbol{Z}, t)$ be the distribution function in a general phase space denoted by $\boldsymbol{Z}$, and f obeys the following collisionless kinetic equation

$$\frac{d}{dt} f = \left(\frac{\partial}{\partial t} + \dot{\boldsymbol{Z}} \cdot \boldsymbol{\nabla}_{\boldsymbol{Z}} \right) f = 0 \ , \tag{1}$$

where

$$\dot{\boldsymbol{Z}} = \boldsymbol{J}(\boldsymbol{Z}, t) \ , \tag{2}$$

and $\boldsymbol{\nabla}_{\boldsymbol{Z}} = \partial/\partial \boldsymbol{Z}$. Let $f = f_0 + \delta f$ with f_0 and δf being, respectively, the equilibrium and perturbed distribution function. Meanwhile, $\boldsymbol{J} = \boldsymbol{J}_0 + \delta \boldsymbol{J}$ with $\boldsymbol{J}_0$ and $\delta \boldsymbol{J}$ corresponding, respectively, to the unperturbed and perturbed phase space orbits. Equation (1) then becomes

$$\frac{d}{dt} f = \left(\left. \frac{d}{dt} \right|_0 + \delta \boldsymbol{J} \cdot \boldsymbol{\nabla}_{\boldsymbol{Z}} \right) (f_0 + \delta f) = 0 \ , \tag{3}$$

where

$$\left.\frac{d}{dt}\right|_0 = \frac{\partial}{\partial t} + \boldsymbol{J}_0 \cdot \boldsymbol{\nabla}_{\boldsymbol{Z}} \; . \tag{4}$$

Noting that $d/dt|_0 f_0 = 0$, Eq. (3) reduces to

$$\frac{d}{dt}\delta f = -\delta \boldsymbol{J} \cdot \boldsymbol{\nabla}_{\boldsymbol{Z}} f_0 \equiv \delta S(\boldsymbol{Z}, t) \; . \tag{5}$$

δf in Eq. (5) can be formally solved via the following oscillating-center transformation defined as

$$\delta f = \exp\left(-\delta \boldsymbol{Z} \cdot \boldsymbol{\nabla}_{\boldsymbol{Z}}\right) \delta f_{\rm OSC} \; , \tag{6}$$

where

$$\frac{d}{dt}\delta \boldsymbol{Z} = \delta \boldsymbol{J} \; . \tag{7}$$

Substituting Eqs. (6) and (7) into Eq. (5), one readily obtains

$$\left.\frac{d}{dt}\right|_0 \delta f_{\rm OSC} = \exp\left(\delta \boldsymbol{Z} \cdot \boldsymbol{\nabla}_{\boldsymbol{Z}}\right) \delta S(\boldsymbol{Z}, t) \; . \tag{8}$$

$\delta f_{\rm OSC}$ can then be symbolically solved via integration along the unperturbed phase space orbit; *i.e.*,

$$\delta f_{\rm OSC} = \left(\left.\frac{d}{dt}\right|_0 \right)^{-1} \exp\left(\delta \boldsymbol{Z} \cdot \boldsymbol{\nabla}_{\boldsymbol{Z}}\right) \delta S(\boldsymbol{Z}, t) \; . \tag{9}$$

Equation (9) simply states that the solution corresponds to integration along the exact (unperturbed plus the perturbed) phase space orbits. In the $|\delta \boldsymbol{Z} \cdot \boldsymbol{\nabla}_{\boldsymbol{Z}}| \to 0^+$ limit, one, as expected, recovers the linear limit.

Equation (9) remains a formal solution, since $\delta \boldsymbol{Z}$ needs to be solved from the nonlinear equation, Eq. (7). For particles, which do not satisfy the linear wave-particle resonance condition, however, $\delta \boldsymbol{Z}$ may be solved via two-time-scale expansions. That is, letting $\delta \boldsymbol{Z} = \delta \overline{\boldsymbol{Z}} + \delta \tilde{\boldsymbol{Z}}$ where $\overline{[...]}$ denotes averaging over an appropriate oscillation period, Eq. (7) than yields

$$\left.\frac{d}{dt}\right|_0 \delta \tilde{\boldsymbol{Z}} = \delta \tilde{\boldsymbol{J}} - \delta \tilde{\boldsymbol{J}} \cdot \boldsymbol{\nabla}_{\rm Z} \delta \overline{\boldsymbol{Z}} - \left[\delta \tilde{\boldsymbol{J}} \cdot \boldsymbol{\nabla}_{\rm Z} \delta \tilde{\boldsymbol{Z}} - \overline{\delta \tilde{\boldsymbol{J}} \cdot \boldsymbol{\nabla}_{\rm Z} \delta \tilde{\boldsymbol{Z}}} \right] \simeq \delta \tilde{\boldsymbol{J}} \; ; \tag{10}$$

and

$$\left.\frac{d}{dt}\right|_0 \delta \overline{\boldsymbol{Z}} = -\overline{\delta \tilde{\boldsymbol{J}} \cdot \boldsymbol{\nabla}_{\rm Z} \delta \tilde{\boldsymbol{Z}}} \; . \tag{11}$$

Here, we have considered $\delta \overline{\boldsymbol{J}} = 0$ in the absence of linear (primary) resonances, and explicitly showed the approximation involved in the small amplitude perturbation expansion underlying the adopted two-time-scale expansion. For our present considerations as well as practical applications, $\delta \tilde{\boldsymbol{Z}}$ given by the leading order Eq. (10) is sufficient.

With $\delta\tilde{\boldsymbol{Z}}$ being the oscillating excursion in the phase space, Eq. (10) can in general be solved via integration along the unperturbed orbits and yields the following expression

$$\delta\tilde{\boldsymbol{Z}} = \sum_{p\in\mathbb{Z}} \delta\boldsymbol{Z}_p \sin\left[\Theta_p(\boldsymbol{Z},t)\right] \ . \tag{12}$$

$\delta f_{\rm OSC}$ of Eq. (9) is then given by

$$\delta f_{\rm OSC} = \left(\left.\frac{d}{dt}\right|_0\right)^{-1} \prod_{p\in\mathbb{Z}} \sum_{\ell\in\mathbb{Z}} J_\ell(\lambda_p) e^{i\ell\Theta_p(\boldsymbol{Z},t)} \delta S(\boldsymbol{Z},t) \ , \tag{13}$$

where $J_\ell(\lambda_p)$ are Bessel functions and

$$\lambda_p = -i\delta\boldsymbol{Z}_p \cdot \boldsymbol{\nabla}_{\boldsymbol{Z}} \ . \tag{14}$$

Equation (13) clearly indicates that, with finite-amplitude fluctuations, λ_p is finite and $\ell \neq 0$ terms will contribute to the $\ell\Theta_p(\boldsymbol{Z},t)$ nonlinear modification to the wave-particle interaction phase, and, consequently, nonlinear (higher-order) wave-particle resonances.

With $\delta f_{\rm OSC}$ derived via Eq. (13), we can then perform the inverse oscillating-center transformation, Eq. (6), and obtain δf as

$$\delta f = \prod_{p'\in\mathbb{Z}} \sum_{\ell'\in\mathbb{Z}} J_\ell(\lambda_{p'}) e^{-i\ell'\Theta_{p'}(\boldsymbol{Z},t)} \delta f_{\rm OSC} \ . \tag{15}$$

Focusing on the single-wave response (*i.e.*, neglecting off-diagonal terms in the expansion series), we have, letting $p' = p$ and $\ell' = \ell$,

$$\delta f = \prod_{p\in\mathbb{Z}} \sum_{\ell\in\mathbb{Z}} J_\ell(\lambda_p) e^{-i\ell\Theta_p(\boldsymbol{Z},t)} \left(\left.\frac{d}{dt}\right|_0\right)^{-1} J_\ell(\lambda_p) e^{i\ell\Theta_p(\boldsymbol{Z},t)} \delta S(\boldsymbol{Z},t) \ . \tag{16}$$

Equation (16) is the desired formal solution of δf, which explicitly contains nonlinear wave-particle resonances via the $\ell \neq 0$ terms due to the finite-$|\lambda_p|$ contributions. In the following section, we will calculate δf explicitly for magnetically trapped particles in an axisymmetric tokamak plasma.

3. Low-frequency response of trapped particles in tokamaks

In order to illustrate the general approach considered in the preceding section, we will focus the analysis here to the specific case of low-frequency responses of magnetically trapped particles in an axisymmetric tokamak. The relevant governing equation is then the following nonlinear gyrokinetic equations [15] for the non-adiabatic component of the perturbed distribution function, δg,

$$\mathcal{L}_g \delta g = -\frac{e}{M} Q F_0 \delta H \ , \tag{17}$$

where e and M represent electric charge and mass of the particles, respectively, while

$$\mathcal{L}_g = \partial_t + v_\parallel \boldsymbol{b}_0 \cdot \boldsymbol{\nabla} + \boldsymbol{v}_d \cdot \boldsymbol{\nabla} + \delta\dot{\boldsymbol{X}} \cdot \boldsymbol{\nabla} \tag{18}$$

$$\equiv \mathcal{L}_{g0} + \delta\dot{\boldsymbol{X}} \cdot \boldsymbol{\nabla} \ , \tag{19}$$

$v_\parallel$ denotes the particle velocity along the equilibrium $\boldsymbol{B}_0$ field, identified by the unit vector $\boldsymbol{b}_0 = \boldsymbol{B}_0/B_0$, $\boldsymbol{v}_d$ is the magnetic gradient and curvature drift,

$$QF_0 = \left(\frac{\partial F_0}{\partial \mathcal{E}} \frac{\partial}{\partial t} + \frac{1}{\Omega} \boldsymbol{\nabla} F_0 \times \boldsymbol{b}_0 \cdot \boldsymbol{\nabla} \right) \ , \tag{20}$$

$\mathcal{E} = v^2/2$ is the energy per unit mass, $\Omega = eB_0/(Mc)$ is the cyclotron frequency,

$$\delta H = \left\langle \delta\phi - \frac{\boldsymbol{v}}{c} \cdot \delta \boldsymbol{A} \right\rangle_\alpha \ , \tag{21}$$

and

$$\delta \dot{\boldsymbol{X}} = \frac{c}{B_0} \boldsymbol{b}_0 \times \boldsymbol{\nabla} \delta H. \tag{22}$$

In Eq. (21), $\langle \ldots \rangle_\alpha$ denotes gyro-averaging; that is, integration along the particle cyclotron motion in the equilibrium magnetic field $\boldsymbol{B}_0$. $\mathcal{L}_g$ of Eq. (18) is, thus, the exact "propagator"‡ in the guiding-center phase space; while $\mathcal{L}_{g0}$ is that along the unperturbed orbit. Equations (18) and (22) define the necessary oscillating-center transformation. Thus, in the lowest order, we have, recalling the notation introduced in Sec. 2 for Eq. (10),

$$\delta \tilde{\boldsymbol{X}} \simeq \delta \tilde{\boldsymbol{X}}_0 \ , \tag{23}$$

and

$$\mathcal{L}_{g0} \delta \tilde{\boldsymbol{X}}_0 \simeq \frac{c}{B_0} \boldsymbol{b}_0 \times \boldsymbol{\nabla} \delta \tilde{H} \ . \tag{24}$$

Let us further assume that the fluctuations be of a single toroidal mode number n, and a single poloidal mode number m. That is

$$\delta \tilde{H} = \frac{1}{2} \left[e^{in\zeta - im\theta - i\omega t} \delta H_m(r) + c.c. \right] \ , \tag{25}$$

where *c.c.* stands for complex conjugate. Letting $\boldsymbol{B}_0 = \boldsymbol{\nabla}\xi \times \boldsymbol{\nabla}\psi$ and $\xi = \zeta - q\theta$, with (r, θ, ζ) being straight magnetic field line toroidal flux coordinates (cf. Appendix A), and defining

$$\delta \tilde{\boldsymbol{X}}_0 = \frac{1}{2} \left[\delta \tilde{\boldsymbol{X}}_{0m} + c.c. \right] \ , \tag{26}$$

Eq. (24) then leads to

$$\begin{aligned} \left(-i\omega + v_\parallel \nabla_\parallel + \boldsymbol{v}_d \cdot \boldsymbol{\nabla} \right) \delta \tilde{\boldsymbol{X}}_{0m} &= \\ &= i \frac{c}{B_0} \left(\boldsymbol{b}_0 \times \boldsymbol{k}_m \right) e^{in\xi - i\omega t + i(nq-m)\theta} \delta H_m \ . \end{aligned} \tag{27}$$

Here, $i\boldsymbol{k}_m = in\boldsymbol{\nabla}\xi + \boldsymbol{\nabla} r \partial_r$. $\delta \tilde{\boldsymbol{X}}_{0m}$ of Eq. (27) can be readily solved as in the case of linear theory [6, 7], summarized in Appendix A. Thus, letting $\boldsymbol{v}_d = \bar{\boldsymbol{v}}_d + \tilde{\boldsymbol{v}}_d$, with $\tilde{\boldsymbol{v}}_d$ being periodic in the magnetic trapping time $\tau_b = 2\pi/\omega_b$ and $v_\parallel \nabla_\parallel \tilde{\boldsymbol{\rho}}_b = \tilde{\boldsymbol{v}}_d$, Eq. (27) becomes

$$\begin{aligned} \left(-i\omega + \partial_{\tau(\theta)} + in\bar{\omega}_d \right) \delta \tilde{\boldsymbol{X}}_{bm} &= \\ &= i \frac{c}{B_0} e^{\tilde{\boldsymbol{\rho}}_b \cdot \boldsymbol{\nabla}} \left(\boldsymbol{b}_0 \times \boldsymbol{k}_m \right) e^{in\xi - i\omega t + i(nq-m)\theta} \delta H_m \ , \end{aligned} \tag{28}$$

‡ Incidentally, we note that, in the standard terminology, propagator is the inverse (integral) operator $\mathcal{L}_g^{-1}$.

where $\delta\tilde{\boldsymbol{X}}_{0m} \equiv e^{-\tilde{\boldsymbol{\rho}}_b\cdot\boldsymbol{\nabla}}\delta\tilde{\boldsymbol{X}}_{bm}$ and $\tau(\theta) = \int_0^\theta d\ell'/v_\parallel(\ell')$ is the time it takes for the particle to move along the unperturbed trajectory, following the $\boldsymbol{B}_0$ magnetic field along the arc-length element $d\ell$. Neglecting the finite-banana-width term, $\tilde{\boldsymbol{\rho}}_b\cdot\boldsymbol{\nabla}$, for now to simplify the presentation and expanding the Fourier series in τ, we have, after straightforward algebra,

$$\delta\tilde{\boldsymbol{X}}_0 \simeq \frac{1}{2}\left[\sum_{p\in\mathbb{Z}}\delta\tilde{\boldsymbol{X}}_{0p}e^{i\Theta_p} + c.c.\right] , \tag{29}$$

where

$$\delta\tilde{\boldsymbol{X}}_{0p} = \frac{c}{B_0}\lambda_{pm}\frac{(\boldsymbol{b}_0\times\boldsymbol{k}_m)\,\delta H_m}{n\bar{\omega}_d - \omega - p\omega_b} , \tag{30}$$

$$\lambda_{pm} = \frac{1}{\tau_b}\oint d\tau e^{ip\omega_b\tau + i(nq-m)\theta} , \tag{31}$$

and

$$\Theta_p = n\xi - \omega t - p\omega_b\tau(\theta) . \tag{32}$$

Note that the integral in Eq. (31) is taken along the unperturbed particle orbits. The effects of finite-banana-width can be readily restored in the present treatment as illustrated in Appendix A, where interested readers can find all necessary details.

To proceed further analytically, let us assume that the perturbed orbit of the magnetically trapped particle is dominated by the $p = p_0$ bounce harmonic; *i.e.*,

$$\delta\tilde{\boldsymbol{X}}_0 \simeq \delta\boldsymbol{X}_{p_0}\sin\Theta_{p_0} . \tag{33}$$

Here, for the sake of simplicity of notation, we have let $\delta\boldsymbol{X}_{p_0} = \delta\tilde{\boldsymbol{X}}_{0p_0}$ and shifted $\Theta_{p_0} \to \Theta_{p_0} - \pi/2$ without loss of generality. The oscillating-center transformation, corresponding to Eq. (6), is then given by

$$\delta g = \exp\left(-\delta\tilde{\boldsymbol{X}}_0\cdot\boldsymbol{\nabla}\right)\delta g_{\rm OSC} , \tag{34}$$

and $\delta g_{\rm OSC}$ satisfies, according to Eqs. (8) and (17)

$$\mathcal{L}_{g0}\delta g_{\rm OSC} = -\frac{e}{M}\exp\left(\delta\tilde{\boldsymbol{X}}_0\cdot\boldsymbol{\nabla}\right)Q_nF_0\delta\tilde{H} . \tag{35}$$

Here,

$$Q_nF_0 = -i\omega\partial_\mathcal{E}F_0 + in\boldsymbol{\nabla}\xi\cdot\boldsymbol{\nabla}F_0\times\boldsymbol{b}_0/\Omega .$$

Equation (35) can be further expressed as

$$\begin{aligned}\mathcal{L}_{g0}\delta g_{\rm OSC} &= \frac{1}{2}\left[\sum_{\ell\in\mathbb{Z}}J_\ell(-i\delta\boldsymbol{X}_{p_0}\cdot\boldsymbol{\nabla})e^{i\ell\Theta_{p_0}}\right.\\ &\qquad\left.\times\left(-\frac{e}{M}\right)Q_nF_0e^{in\xi - i\omega t + i(nq-m)\theta}\delta H_m + c.c.\right]\\ &= \left(-\frac{e}{M}\right)\frac{1}{2}\left[\sum_{\ell\in\mathbb{Z}}\sum_{p\in\mathbb{Z}}J_\ell(-i\delta\boldsymbol{X}_{p_0}\cdot\boldsymbol{\nabla})Q_nF_0\right.\\ &\qquad\left.\times\lambda_{pm}\delta H_me^{i(\ell+1)(n\bar{\omega}_d\tau - \omega t) - i(p+\ell p_0)\omega_b\tau} + c.c.\right] ,\end{aligned} \tag{36}$$

which yields

$$\delta g_{\rm OSC} = \frac{1}{2}\left[i\left(\frac{e}{M}\right) \sum_{\ell\in\mathbb{Z}} \sum_{p\in\mathbb{Z}} \frac{Q_n F_0}{[(\ell+1)(n\bar{\omega}_d - \omega) - (p+\ell p_0)\omega_b]} \right.$$
$$\left. \times\ J_\ell(-i\delta \boldsymbol{X}_{p_0}\cdot\boldsymbol{\nabla})\lambda_{pm}\delta H_m e^{i\Theta_{\ell,p}} + c.c. \right] \ , \qquad (37)$$

and

$$\Theta_{\ell,p} = (\ell+1)(n\xi - \omega t) - (p+\ell p_0)\omega_b \tau \ . \qquad (38)$$

We note that Eq. (37) clearly exhibits that, due to the $\ell \neq 0$ nonlinear term (thus, ℓ is the nonlinear harmonic), the following nonlinear resonance condition applies

$$\frac{\omega_b}{n\bar{\omega}_d - \omega} = \frac{\ell+1}{p+\ell p_0} \qquad (39)$$

or, setting the nonlinear bounce harmonic $p = p_0 + p'$,

$$\frac{\omega_b}{n\bar{\omega}_d - \omega} = \frac{\ell+1}{(\ell+1)p_0 + p'} \ . \qquad (40)$$

Equations (39) and (40) are the same as Eqs. (A.13) and (A.14), derived in Appendix A within a more general theoretical framework.

Performing the inverse oscillating-center transformation, Eq. (34), and focusing on the single-ω and single-n response, we then obtain the desired δg, which includes nonlinear effects due to the perturbed phase space orbits for linearly non-resonant particles;

$$\delta g \simeq \frac{1}{2}\left[i\left(\frac{e}{M}\right) \sum_{\ell\in\mathbb{Z}} \sum_{p\in\mathbb{Z}} \frac{Q_n F_0}{[(\ell+1)(n\bar{\omega}_d - \omega) - (p+\ell p_0)\omega_b]} \right.$$
$$\left. \times\ \left|J_\ell(-i\delta \boldsymbol{X}_{p_0}\cdot\boldsymbol{\nabla})\right|^2 \lambda_{pm}\delta H_m e^{i\Theta_p} + c.c. \right] \ . \qquad (41)$$

Let us remark that the current theoretical analysis and results assume that the particles are linearly non-resonant and their phase space motion is periodic, *e.g.* Eq. (33), with a period much shorter than the appropriate nonlinear time scale. For a monochromatic wave with a nearly constant amplitude, this assumption is valid if

$$|p_0\omega_b + \omega - n\bar{\omega}_d| \gg |\omega_{\rm tr}| \ , \qquad (42)$$

where $\omega_{\rm tr}$ is the nonlinear bounce frequency of the particles trapped by the wave. Appendix A provides a more thorough and systematic analysis on the effects of perturbed phase space orbits on the wave-particle interaction phase and the corresponding nonlinear wave-particle resonance conditions. For details, we refer interested readers to that analysis. Here, we merely note that Eq. (42) can be considered to be well satisfied for the problem of interest.

As a possible application of the present theoretical result, δg given by Eq. (41) can be used to calculate the quasilinear particle flux induced by the nonlinear wave-particle flux. Following Chen [3] and Chen and Zonca [16], it is then straightforward to derive

$$\frac{\partial}{\partial t}\left[N\right]_s + \frac{1}{V'_\psi}\frac{\partial}{\partial\psi}\left(V'_\psi \Gamma^{n\ell}_{\psi,t}\right) = 0 \ , \qquad (43)$$

where $[N]_s$ is the magnetic flux surface averaged density, $V'_\psi = \oint d\ell/B_0$, and

$$\Gamma^{n\ell}_{\psi,t} = -\frac{\pi}{2}ec\sum_{\ell\in\mathbb{Z}}\sum_{p\in\mathbb{Z}} \int \Bigg[d\boldsymbol{v}\delta\Big((\ell+1)(n\bar{\omega}_d - \omega) - (p+\ell p_0)\omega_b\Big) \times \left|J_\ell(-i\delta\boldsymbol{X}_{p_0}\cdot\boldsymbol{\nabla})\right|^2 |\lambda_{pm}|^2 |\delta H_m|^2 \left(-\frac{n\omega}{M}\frac{\partial F_0}{\partial\mathcal{E}} + \frac{n^2 c}{e}\frac{\partial F_0}{\partial\psi}\right)\Bigg] . \quad (44)$$

Note that here, we have adopted the same normalizations of the fluctuation spectrum in $(\omega, \boldsymbol{k})$ space introduced by [3]. The superscript $n\ell$ denotes that the transport, in addition to the usual quasilinear response via the $\ell = 0$ wave-particle resonances, also includes nonlinear wave-particle resonances via $\ell \neq 0$. Equation (44) can, of course, be extended to include a spectrum of toroidal (n) and poloidal (m) modes; as well as be generalized to the transports of parallel momentum and energy [3, 7, 16].

4. Summary and Discussions

In this work, we have developed a systematic general theoretical approach for analyzing self-consistent plasma dynamics including the effects of nonlinear wave-particle resonances. Our theoretical approach is based on the scheme of the oscillating-center transformation. As a specific illustration of this general approach, the kinetic response of magnetically trapped particles to low-frequency fluctuations in axisymmetric tokamaks is derived via the nonlinear gyrokinetic equations. The derived perturbed non-adiabatic distribution function can, as an example, be used to derive the corresponding quasilinear particle flux induced by the nonlinear wave-particle resonances in addition to the usual linear wave-particle resonances. Quantitative estimates of the transport coefficients remain, however, to be carried out.

The present theoretical analysis has, obviously, broader applications than those analyzed here. For example, the present analysis could be easily extended to the case of circulating particles. Furthermore, nonlinear wave-particle resonances can be expected to play some roles in the longer time-scale evolution of instabilities as the wave-particle exchange of energy and momentum via the linear resonances diminish and fluctuations grow to some significant finite amplitudes. This could, in principle, be done by applying δg of Eq. (41) to the gyrokinetic vorticity equation [7, 17, 18, 19]. All these anticipations/conjectures are worthy of further investigations by, preferably, numerical simulations.

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Appendix A. Wave-particle resonances in tokamaks

Let us consider $\delta S(\boldsymbol{Z}, t)$, introduced in Eq. (5), and follow the approach outlined in Sec. 2 based on systematic perturbation expansions, which ultimately allow us to express physical solutions by integration along unperturbed orbits. Following Refs. [6, 7] and using straight magnetic field line toroidal flux coordinates (r, θ, ζ), magnetically trapped particle trajectories in tokamaks can be represented as

$$\begin{aligned} r &= \overline{r}(E, \mu, P_\phi) + \tilde{\rho}_c \left(\theta_c; E, \mu, P_\phi\right) \ , \\ \theta &= \tilde{\Theta}_c \left(\theta_c; E, \mu, P_\phi\right) \ , \\ \zeta &= \zeta_c + \overline{q}(E, \mu, P_\phi) \tilde{\Theta}_c \left(\theta_c; E, \mu, P_\phi\right) + \tilde{\Xi}_c \left(\theta_c; E, \mu, P_\phi\right) \ , \end{aligned} \tag{A.1}$$

where θ_c and ζ_c are canonical angles that define the (unperturbed) orbital frequencies

$$\begin{aligned} \dot{\theta}_c &= \omega_b(E, \mu, P_\phi) \ , \\ \dot{\zeta}_c &= \overline{\omega}_d(E, \mu, P_\phi) \ , \end{aligned} \tag{A.2}$$

as a function of the constants of motion E, the energy, μ, the magnetic moment, and P_ϕ the canonical toroidal angular momentum. Furthermore, with the usual definition of the safety factor

$$q(r) = \frac{\boldsymbol{B}_0 \cdot \boldsymbol{\nabla} \zeta}{\boldsymbol{B}_0 \cdot \boldsymbol{\nabla} \theta} \ , \tag{A.3}$$

the orbit averaged $\overline{q}(E, \mu, P_\phi)$ is obtained as

$$\overline{q}\left(E, \mu, P_\phi\right) = \frac{\oint q(r) d\theta}{\oint d\theta} = \frac{\oint q(\overline{r} + \tilde{\rho}_c) \dot{\theta} d\theta_c}{\oint \dot{\theta} d\theta_c} \ . \tag{A.4}$$

Finally, $\tilde{\rho}_c\left(\theta_c; E, \mu, P_\phi\right)$, $\tilde{\Theta}_c\left(\theta_c; E, \mu, P_\phi\right)$ and $\tilde{\Xi}_c\left(\theta_c; E, \mu, P_\phi\right)$ are periodic functions of θ_c that are parameterized by E, μ, P_ϕ. Similar to Eq. (A.1), circulating particle orbits can also be conveniently represented [6, 7] but their explicit parameterization is not needed here. Given this representation of magnetically trapped particles unperturbed motion in tokamaks, $\delta S(\boldsymbol{Z}, t)$ along the particle orbit is readily represented as

$$\begin{aligned} \delta S(\boldsymbol{Z}, t) &= e^{-i\omega t + in\zeta} \sum_{m \in \mathbb{Z}} e^{-im\theta} \delta \mathcal{S}_m \left(r; E, \mu, P_\phi\right) \\ &= e^{-i\omega t + in\zeta_c} \sum_{m,p \in \mathbb{Z}} e^{-ip\theta_c} \delta \mathcal{S}_{m,p} \left(\overline{r}; E, \mu, P_\phi\right) \ , \end{aligned} \tag{A.5}$$

where

$$\delta \mathcal{S}_{m,p}\left(\overline{r}; E, \mu, P_\phi\right) = \frac{1}{2\pi} \oint d\theta_c e^{in\tilde{\Xi}_c + i(n\overline{q} - m)\tilde{\Theta}_c + ip\theta_c} \delta \mathcal{S}_m \left(\overline{r} + \tilde{\rho}_c; E, \mu, P_\phi\right) \ . \tag{A.6}$$

Here, for the sake of simplicity, we assumed an imposed perturbation with given frequency ω and toroidal mode number n. Equation (A.5) readily shows that linear wave-particle resonance occurs for

$$\omega = n\overline{\omega}_d(E, \mu, P_\phi) - p\omega_b(E, \mu, P_\phi) \ . \tag{A.7}$$

If the linear resonance condition, Eq. (A.7), is not satisfied (cf. Sec. 2) but particle actions (constants of the unperturbed motion) are still in the neighborhood of the linear resonance identified by a given $p = p_0$, we can adopt the standard approach of secular perturbation theory [20, 21] by transforming to the wave frame, where θ and $\eta_{p_0}(\boldsymbol{Z}, t)$,

$$\dot{\eta}_{p_0}(\boldsymbol{Z}, t) = n\overline{\omega}_d(E, \mu, P_\phi) - p_0\omega_b(E, \mu, P_\phi) - \omega \ , \tag{A.8}$$

provide the new angles of a system with two degrees of freedom. Since $|\dot{\eta}_{p_0}| \ll |\dot{\theta}|$, it is possible to average over the fast θ dependences, and the averaged Hamiltonian will become the "standard Hamiltonian" [20, 21] providing the universal description of the motion near an isolated resonance [22].

In general, it is possible to transform to action-angle coordinates of the particle motion near the isolated resonance $\dot{\eta}_{p_0}(\boldsymbol{Z}, t) = 0$. More specifically,

$$\eta_{p_0}(\boldsymbol{Z}, t) = \phi_{p_0}\left(\Theta_{p_0}(\boldsymbol{Z}, t)\right) \ , \tag{A.9}$$

where $\phi_{p_0}(\Theta_{p_0})$ is a periodic function of the angle Θ_{p_0}, for which

$$\dot{\Theta}_{p_0}(\boldsymbol{Z}, t) = \omega_{\rm tr}(I_{\Theta_{p_0}}) \tag{A.10}$$

is the wave-particle trapping frequency. In Eq. (A.10), for the sake of simplicity, we only reported the dependence of the trapping frequency on the action $I_{\Theta_{p_0}}$, conjugate to the angle Θ_{p_0}, leaving implicit the parametric dependences on the constants of the unperturbed motion identifying the primary (linear) resonance through Eq. (A.7) for $p = p_0$. Secondary (nonlinear) resonances account for resonances between the perturbed motion near the considered isolated resonance and the periodic motion in the other (θ) degree of freedom. To see this, let us consider the particle dynamics near an elliptic fixed point of the primary libration. Then, Eq. (A.9) can be approximated as

$$\eta_{p_0}(\boldsymbol{Z}, t) \simeq \Delta\eta_{p_0}(I_{\Theta_{p_0}}) \sin\left(\Theta_{p_0}(\boldsymbol{Z}, t)\right) \ . \tag{A.11}$$

Similarly, we can express the periodic nonlinear distortions of phase space trajectories and obtain the representation of Eq. (12) in Sec. 2, where, for simplicity, we have reabsorbed any possibly needed phase shift into the definition of Θ_{p_0} (cf. also Eq. (33) in Section 3). As in Eq. (A.11), in general, $\Delta\eta_{p_0} = \mathcal{O}(1)$, we generally have $\lambda_{p_0} = \mathcal{O}(1)$ in Eq. (14). Thus, Eq. (13) can be specialized to $p = p_0$ and yields, together with Eq. (A.5)

$$\delta f_{\rm OSC} = \left(\left.\frac{d}{dt}\right|_0 \right)^{-1} \sum_{\ell \in \mathbb{Z}} J_\ell(\lambda_{p_0}) \times e^{i\ell\Theta_{p_0}} e^{-i\omega t + in\zeta_c} \sum_{m,p \in \mathbb{Z}} e^{-ip\theta_c} \delta\mathcal{S}_{m,p}\left(\overline{r}; E, \mu, P_\phi\right) \ . \tag{A.12}$$

The nonlinear resonance condition, generalizing Eq. (A.7), then becomes

$$\omega = n\overline{\omega}_d(E, \mu, P_\phi) - p\omega_b(E, \mu, P_\phi) + \ell\omega_{\rm tr}(I_{\Theta_{p_0}}) \ . \tag{A.13}$$

Noting Eqs. (A.2) and (A.10), we can generally conclude that $\ell = \mathcal{O}(\omega/\omega_{\rm tr}) = \mathcal{O}(|\boldsymbol{B}_0|^{1/2}/|\delta\boldsymbol{B}|^{1/2}) \gg 1$ [20, 21]. Meanwhile, the contribution of the considered nonlinear resonance is weighed by $J_\ell(\lambda_{p_0}) = \mathcal{O}((\lambda_{p_0}/2)^\ell/\ell!) \ll 1$ [20, 21].

Secondary (nonlinear) resonances also account for the modification of the resonance condition due to the presence of finite amplitude fluctuations when the linear resonance, Eq. (A.7), is not nearly satisfied. In this case, $\Theta_{p_0}(\boldsymbol{Z},t) = \eta_{p_0}(\boldsymbol{Z},t)$ in Eq. (A.9), expressing the leading order unperturbed motion, while the nonlinear distortions of phase space trajectories are expressed as periodic functions of $\Theta_{p_0}(\boldsymbol{Z},t) = \eta_{p_0}(\boldsymbol{Z},t)$, *e.g.* as in Eqs. (12) and (33), with $\lambda_p = \mathcal{O}(|\delta\boldsymbol{B}|/|\boldsymbol{B}_0|) \ll 1$. Equation (13), specialized to $p = p_0$, still holds in the form of Eq. (A.12), where now $\dot{\Theta}_{p_0} = n\overline{\omega}_d(E,\mu,P_\phi) - p_0\omega_b(E,\mu,P_\phi) - \omega$ due to Eq. (A.8). Thus, the nonlinear resonance condition reads

$$\omega(1+\ell) = n(1+\ell)\overline{\omega}_d(E,\mu,P_\phi) - p_0(1+\ell)\omega_b(E,\mu,P_\phi) - p'\omega_b(E,\mu,P_\phi) \;, \tag{A.14}$$

where n is the toroidal harmonic number of the considered fluctuation, p_0 is the primary (linear) bounce harmonic, $p = p_0 + p'$ is the nonlinear bounce harmonic, and ℓ is the nonlinear harmonic [10, 11, 12, 13, 14]. Clearly, Eqs. (A.13) and (A.14) are entirely consistent when $\omega_{\rm tr}(I_{\Theta_{p_0}}) = n\overline{\omega}_d(E,\mu,P_\phi) - p_0\omega_b(E,\mu,P_\phi) - \omega$, which is intuitive for the considered resonant interaction.

The above analysis can be straightforwardly extended to the case of circulating particles [6, 7]. The corresponding expressions for the nonlinear wave-particle resonance conditions are then given, respectively, by

$$\begin{aligned}\omega = {} & n\overline{\omega}_d(E,\mu,P_\phi) + (n\overline{q}(E,\mu,P_\phi) - m)\,\omega_t(E,\mu,P_\phi) \\ & - p\omega_t(E,\mu,P_\phi) + \ell\omega_{\rm tr}(I_{\Theta_{p_0}}) \;,\end{aligned} \tag{A.15}$$

which is the analogue of Eq. (A.13), and

$$\begin{aligned}\omega(1+\ell) = {} & n(1+\ell)\overline{\omega}_d(E,\mu,P_\phi) + (n\overline{q}(E,\mu,P_\phi) - m)\,(1+\ell)\omega_t(E,\mu,P_\phi) \\ & - p_0(1+\ell)\omega_t(E,\mu,P_\phi) - p'\omega_t(E,\mu,P_\phi) \;,\end{aligned} \tag{A.16}$$

which corresponds to Eq. (A.14). In Eqs. (A.15) and (A.16), $\omega_t(E,\mu,P_\phi)$ is the circulating particle transit frequency, while $\overline{q}(E,\mu,P_\phi)$ is defined in Eq. (A.4). Meanwhile, the interpretations of n (m) as the toroidal (poloidal) harmonic number of the considered fluctuation, of p_0 as the primary (linear) bounce harmonic, of $p = p_0 + p'$ as the nonlinear bounce harmonic, and of ℓ as the nonlinear harmonic remain the same as above [10, 11, 12, 13, 14].

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