Nonlinear reversed shear Alfvén eigenmode saturation due to spontaneous zonal current generation

Shizhao Wei[1], **Tao Wang**[1], **Ningfei Chen**[1] **and Zhiyong Qiu**[1,2]‡

[1]Institute for Fusion Theory and Simulation and Department of Physics, Zhejiang University, Hangzhou 310027, People's Republic of China

[2] Center for Nonlinear Plasma Science and ENEA C. R. Frascati, Frascati, Italy

Abstract. General nonlinear equations describing reversed shear Alfvén eigenmode (RSAE) self-modulation via zero frequency zonal structure (ZFZS) generation are derived using nonlinear gyrokinetic theory, which are then applied to study the spontaneous ZFZS excitation as well as RSAE nonlinear saturation. It is found that both electrostatic zonal flow (ZF) and electromagnetic zonal current (ZC) can be preferentially excited by finite amplitude RSAE, depending on specific plasma parameters. The modification to local shear Alfvén wave continuum is evaluated using the derived saturation level of ZC, which is shown to play comparable role in saturating RSAE with the ZFZS scattering.

Keywords: wave-wave interaction, reversed shear Alfvén eigenmode, zonal current, modulational instability, nonlinear saturation

‡ Corresponding author. E-mail: zqiu@zju.edu.cn.

1. Introduction

The future tokamak-based fusion power plants and experimental devices towards this goal (e.g., the ITER [1]), are expected to operate at steady state with a substantial non-inductive current fraction [2], which generally renders a reversed shear scenario [3]. In this circumstance, the reversed shear Alfvén eigenmode (RSAE, also known as Alfvén cascade) is frequently observed [4] as driven unstable by the energetic particles (EPs) [5, 6, 7, 8] in present day tokamaks, and is expected to play significant roles in future reactors [9], in transporting fusion alpha particles to tokamak edge [10], which has a deleterious effects on plasma self-heating, and may damage the plasma facing components [11]. RSAE is a branch of shear Alfvén wave (SAW) eigenmode localized radially near the minimum of the safety factor q profile [12], which is denoted as $q_{\rm min}$. The lowest order RSAE frequency in the incompressible limit, $\omega \simeq k_\parallel v_{\rm A} \simeq |n - m/q_{\rm min}|v_{\rm A}/R$, reflects the sensitive dependence on the instantaneous $q_{\rm min}$ value for given toroidal/poloidal mode numbers n/m, and this feature can be used in q-profile measurement, i.e., MHD spectroscopy [13, 6]. Here, $k_\parallel$ is the wave number parallel to the equilibrium magnetic field $\mathbf{B}$, $v_{\rm A}$ is the Alfvén speed, and R is the major radius. RSAEs are typically dominated by a poloidal harmonic, with the radial width $\propto \sqrt{q/(r_0^2 q'')}$ with r_0 being the radial location of $q_{\rm min}$, and $q'' \equiv \partial_r^2 q$. Despite the fairly good understanding of these linear physics, the nonlinear dynamics of RSAE still attracts recent research interest, especially in view of the EP as well as thermal particle transport induced by the associated electromagnetic field perturbations [14, 10, 15, 16]. The transport rate is closely related with the perturbation amplitude [17, 18]; and in reactor relevant cases with many SAWs simultaneously driven unstable by EPs, the EP orbit could become chaotic and eventually lost in the presence of many low amplitude SAWs (threshold value $\delta B/B \sim \mathcal{O}(10^{-4})$) [19, 20]. Thus, the assessment of the nonlinear RSAE saturation mechanism and amplitude plays a crucial role in evaluating the operation scenario and the EP confinement property.

In general, the channels of SAW nonlinear saturation can be classified into two routes, namely, the wave-particle nonlinear interaction and wave-wave nonlinear couplings [8, 21, 22, 23]. The former focuses on the perturbation to the resonant EP phase space distribution function by finite amplitude SAWs [22, 23], and is widely investigated by numerical simulations, as reviewed in Refs. [8, 24]. By contrast, the latter is relatively less explored. Most of the previous analytical works consider the toroidal Alfvén eigenmode (TAE) [25] as a paradigm case, including the saturation via ion induced scattering [26, 27], nonlinear modification to the SAW continuum structure [28, 29], the spontaneous generation of zero frequency zonal structures (ZFZS) [30, 31, 32] as well as geodesic acoustic mode (GAM) [33, 34]. Since the RSAEs are expected to be prevalent in future steady-state burning plasmas, the RSAE saturation via wave-wave nonlinearity deserves special attention. In particular, the toroidally symmetric zonal field structures [18], including the ZFZS, are well known to play important roles in regulating drift wave turbulences [35, 36, 37, 38] including drift

Alfvén waves, and thus, leading to cross-scale couplings [39] and nonlinear saturation via scattering to short radial wavelength regime (or shearing in some literatures). In this work, spontaneous ZFZS excitation by RSAE modulational instability is analyzed using nonlinear gyrokinetic theory.

As noted above, the spontaneous excitation of ZFZS by TAE is first discussed in [30]. It is shown that under certain conditions, the zonal current (ZC) is preferentially excited over the electrostatic zonal flow (ZF), with the branch ratio of ZF/ZC excitation determined by various geometry effects, including the breaking of pure Alfvénic state by toroidicity and neoclassical shielding of ZF [40]. In contrast to the TAE case, it is shown in Ref. [41] that for beta-induced Alfvén eigenmode (BAE), the excitation of ZF generally dominates, due to the $|k_\parallel v_{\rm A}/\omega| \ll 1$ ordering. For the case of RSAE analyzed herein, its frequency is sensitively determined by the value of $q_{\rm min}$ and the underlying values of toroidal/poloidal mode numbers n/m, and generally sweeps in-between the typical BAE to TAE frequency ranges. It is shown that depending on the specific plasma parameters including $k_\parallel$, both ZC and ZF generation may dominate, and the previous conclusions on TAE [30] and BAE [41] can be recovered as limiting cases of the general nonlinear dispersion relation derived without assuming specific plasma parameters.

We note that, several nonlinear processes, with comparable cross-sections, may be comparably important in saturating AEs, as addressed in Ref. [42]. In particular, a channel unique for RSAE saturation is proposed in this work. Due to the ZC and the associated perturbed poloidal magnetic field generation, the q-profile is modulated, which leads to the modification of the local SAW continuum in the vicinity of $q_{\rm min}$, and consequently RSAE saturation. The relevance of this channel on RSAE nonlinear saturation is analyzed and evaluated.

This paper is arranged as follows. In Sec. 2, the theoretical model is given. In Sec. 3, the generally nonlinear equations describing RSAE evolution and ZFZS excitation are derived. Sec. 4 is devoted to study the linear growth stage of the modulational instability; and the nonlinear saturation of RSAE via ZFZS scattering is investigated in Sec. 5. Finally, a brief conclusion and discussion is given in Sec. 6.

2. Theoretical model

The nonlinear evolution of this system is studied using the standard nonlinear perturbation theory, considering a shifted circular tokamak equilibrium described by a set of field-aligned flux coordinates (r, θ, φ). The perturbed fields are represented by two field variables, namely, the electrostatic potential $\delta\phi$ and the parallel component of vector potential $\delta A_\parallel$, while the parallel magnetic field fluctuation $\delta B_\parallel$ is suppressed, consistent with $\beta \ll 1$ ordering of typical laboratory plasmas. Here, β is the ratio of thermal to magnetic pressures. For convenience, $\delta A_\parallel$ is replaced by $\delta\psi \equiv \omega\delta A_\parallel/(ck_\parallel)$, such that the ideal MHD limit, i.e., vanishing parallel electric field fluctuation $\delta E_\parallel = 0$ corresponds to simply $\delta\phi = \delta\psi$. In this work, it is assumed that the RSAE is excited by a source outside this nonlinear system, such as EPs, and the nonlinear coupling is dominated by bulk plasma contribution. For the cases with EPs contributing significantly to the nonlinear ZFZS generation [43, 44], interested readers may refer to Refs. [31, 32] for more systematic discussion. To start with, we consider the two-field coupled system which consists of a RSAE (subscript 'R') and ZFZS (subscript 'Z'), i.e., $\delta\phi = \delta\phi_R + \delta\phi_Z$ with $\delta\phi_R = \delta\phi_0 + \delta\phi_{0^*}$. Here, $\delta\phi_0$ represents RSAE with positive real frequency and $\delta\phi_{0^*}$ represents the counterpart with negative real frequency, of which there may be a rich spectrum of different radial eigen-states.

Considering the reactor relevant parameter regime with $nq \gg 1$, the ballooning mode representation [45] for RSAE is adopted,

$$\delta\phi_0 = A_0 e^{i(n\phi-\hat{m}\theta-\omega_0 t)} e^{i\int \hat{k}_{r,0}dr} \sum_j e^{-ij\theta}\Phi_0\left(x-j\right) + c.c..$$

Here, $m = \hat{m}+j$ with $\hat{m}$ being the reference poloidal mode number, $x \equiv nq-\hat{m}$, Φ_0 is the parallel mode structure with the typical radial extension comparable to distance between neighboring mode rational surfaces, A_0 is the mode envelope amplitude and $\hat{k}_{r,0}$ is the radial envelope wavenumber accounting for the slowly varying radial structures. Note that, RSAE is typically characterized by one dominant poloidal harmonic, while multiple sub-dominant poloidal harmonics exist due to toroidicity. Furthermore, $\int |\Phi_0|^2 dx = 1$ is used as normalization condition.

Consequently, the ZFZS is expected to have a fine radial structure in addition to the well-known meso-scale structure [38], as a result of the RSAE fine radial mode structure highly localized around q_{min}. For ZFZS dominated by $n = 0, m = 0$ scalar potential perturbation [37], we take

$$\delta\phi_Z = A_Z e^{i\int \hat{k}_Z dr - i\omega_Z t} \sum_j \Phi_Z\left(x-j\right) + c.c.,$$

with Φ_Z accounting for the fine radial structure [41] due to nonlinear mode coupling and $A_Z \exp i\int \hat{k}_Z dr$ being the well-known meso-scale structure. This general representation adopted here can be applied to recover the results obtained from linear growth stage of the modulational instability, by separating RSAE into pump and upper/lower sidebands, as often used in previous papers [37]; and is also recovered in Sec. 4, from the derived general nonlinear equations.

The governing equations describing the nonlinear processes can be derived from quasi-neutrality condition

$$\frac{n_0 e^2}{T_i}\left(1+\frac{T_i}{T_e}\right)\delta\phi_k = \sum_s \left\langle qJ_k\delta H_k\right\rangle_s, \tag{1}$$

and nonlinear gyrokinetic vorticity equation derived from parallel Ampère's law [8]

$$\begin{aligned}&\frac{c^2}{4\pi\omega_k^2}B\frac{\partial}{\partial l}\frac{k_\perp^2}{B}\frac{\partial}{\partial l}\delta\psi_k + \frac{e^2}{T_i}\left\langle\left(1-J_k^2\right)F_0\right\rangle\delta\phi_k - \sum_s\left\langle\frac{q}{\omega_k}J_k\omega_d\delta H_k\right\rangle \\ &= -i\frac{c}{B_0\omega_k}\sum_{\mathbf{k}=\mathbf{k}'+\mathbf{k}''}\hat{\mathbf{b}}\cdot\mathbf{k}''\times\mathbf{k}'\left[\frac{c^2}{4\pi}k_\perp''^2\frac{\partial_l\delta\psi_{k'}\partial_l\delta\psi_{k''}}{\omega_{k'}\omega_{k''}}\right. \\ &\left. + \left\langle e\left(J_kJ_{k'}-J_{k''}\right)\delta L_{k'}\delta H_{k''}\right\rangle\right].\end{aligned} \tag{2}$$

Here, $J_k \equiv J_0(k_\perp \rho)$ with J_0 being the Bessel function of zero index accounting for finite Larmor radius effects, $\rho = v_\perp/\Omega_c$ is the Larmor radius with Ω_c being the cyclotron frequency, F_0 is the equilibrium particle distribution function, $\omega_d = (v_\perp^2 + 2v_\parallel^2)(k_r \sin\theta + k_\theta \cos\theta)/(2\Omega_c R)$ is the magnetic drift frequency, l is the length along the equilibrium magnetic field line, $\langle\cdots\rangle$ means velocity space integration, $\sum_s$ is the summation of different particle species with $s = i, e$ representing ion and electron, and $\delta L_k \equiv \delta\phi_k - k_\parallel v_\parallel \delta\psi_k/\omega_k$. The three terms on the left hand side of Eq. (2) are, respectively, the field line bending, inertial and curvature coupling terms, dominating the linear SAW physics. The two terms on the right hand side of Eq. (2) correspond to Maxwell (MX) and Reynolds stresses (RS) [46] that contribute to nonlinear mode couplings as MX and RS do not cancel each other, with their contribution dominating in the radially fast varying inertial layer [37], and $\sum_{\mathbf{k}=\mathbf{k}'+\mathbf{k}''}$ indicates the wavenumber and frequency matching condition required for nonlinear mode coupling. δH_k is the nonadiabatic particle response, which can be derived from nonlinear gyrokinetic equation [47]:

$$\left(-i\omega_k + v_\parallel \partial_l + i\omega_d\right)\delta H_k = -i\omega_k \frac{q}{T} F_0 J_k \delta L_k$$
$$-\frac{c}{B_0}\sum_{\mathbf{k}=\mathbf{k}'+\mathbf{k}''} \hat{\mathbf{b}}\cdot\mathbf{k}''\times\mathbf{k}' J_{k'}\delta L_{k'}\delta H_{k''}. \tag{3}$$

For RSAE with $|k_\parallel v_e| \gg |\omega_k| \gg |k_\parallel v_i|, |\omega_d|$, the linear ion/electron responses can be derived to the leading order as $\delta H^L_{k,i} = eF_0 J_k \delta\phi_k/T_i$ and $\delta H^L_{k,e} = -eF_0\delta\psi_k/T_e$. Furthermore, one can have, to the leading order, ideal MHD constraint is satisfied, i.e., $\delta\phi_R = \delta\psi_R$, by substituting the ion/electron responses of RSAE into quasi-neutrality condition.

On the other hand, considering such a nonlinear system dominated by SAW instabilities, we can also use the parallel component of the nonlinear ideal Ohm's law as an alternative to Eq. (1),

$$\delta E_{\parallel,k} = -\sum_{\mathbf{k}=\mathbf{k}'+\mathbf{k}''} \hat{\mathbf{b}}\cdot\delta\mathbf{u}_{k'}\times\delta\mathbf{B}_{k''}/c, \tag{4}$$

with $\delta\mathbf{u}$ being the $\mathbf{E}\times\mathbf{B}$ drift velocity. We note that Eq. (4) is equivalent to Eq. (1), ignoring the high order $\mathcal{O}(k_\perp^2\rho_i^2)$ corrections.

3. General nonlinear equations

In this section, the general nonlinear equations describing the self-consistent RSAE evolution are derived, including the generation of ZFZS and the feedback modulation of RSAE by ZFZS. Generally speaking, the nonlinear process can be divided into two stages, i.e., linear growth stage and strongly nonlinear stage, by whether the modulation to the pump wave is small. The governing equations derived in this section,without separating RSAE into pump wave and its sidebands, are general, and can be used for describing both stages as shown in later sections [48, 49].

Considering the nonlinear coupling dominated by the radially fast varying inertial region, one can obtain the equation describing the electrostatic ZF excitation from surface averaged vorticity equation as

$$\omega_Z\hat{\chi}_Z\delta\phi_Z = -i\frac{c}{B}k_\theta\left(1-\frac{k_{\parallel,0}^2v_{\rm A}^2}{\omega_0^2}\right)(k_{r,0}-k_{r,0^*})\left|\delta\phi_0\right|^2. \quad (5)$$

Here, $\hat{\chi}_Z = \chi_Z/(k_Z^2\rho_i^2) \simeq 1.6q^2\epsilon^{-1/2}$ with χ_Z being the well-known neoclassical shielding of ZFZS [40] and $\epsilon \equiv r/R$ being the inverse aspect ratio of the torus. One can note that, $(k_{r,0}-k_{r,0^*})|\delta\Phi_0|^2 \equiv [(\hat{k}_{r,0}-\hat{k}_{r,0^*})-i(\partial_r ln\Phi_0-\partial_r ln\Phi_{0^*})]|\delta\Phi_0|^2$ being radial modulation with $(\hat{k}_{r,0}-\hat{k}_{r,0^*})$ denoting envelope modulation [30] and $(\partial_r ln\Phi_0-\partial_r ln\Phi_{0^*})$ denoting parallel mode structure evolution [41], which gives the fine radial structure of ZFZS. For RSAE typically dominated by one or two poloidal harmonics, $(\partial_r ln\Phi_0-\partial_r ln\Phi_{0^*})$ is the dominant term, and determines the zonal structure radial wavenumber $k_Z = -i(\partial_r\ln\Phi_0-\partial_r\ln\Phi_{0^*})$, as addressed in Ref. [32].

The equation describing the electromagnetic ZC excitation can be derived from Eq. (4), considering $k_{\parallel,Z}=0$ and noting $\delta\psi_Z \equiv \omega_0\delta A_{\parallel,Z}/(ck_{\parallel,0})$ is defined using the frequency and parallel wavenumber of RSAE, as

$$\delta\psi_Z = -i\frac{c}{B}k_{\theta,0}k_Z\frac{1}{\omega_0}\left|\delta\phi_0\right|^2. \quad (6)$$

In deriving Eq.(6), ideal MHD condition for RSAE ($\delta\phi_0=\delta\psi_0$) is used, and $\partial_r\ln\delta\psi_Z = \partial_r\ln|\delta\phi_0|^2$ is noted.

One can also derive the corresponding equations describing RSAE from Eq. (4) as

$$\delta\phi_0-\delta\psi_0 = -i\frac{c}{B}\frac{k_Zk_{\theta,0}}{\omega_0}\delta\phi_0\left(\delta\phi_Z-\delta\psi_Z\right), \quad (7)$$

which describes the deviation from ideal MHD constraint due to nonlinear ZFZS modulation. The other equation for RSAE can be derived from nonlinear vorticity equation as

$$k_{\perp,0}^2\left(-\frac{k_{\parallel,0}^2v_{\rm A}^2}{\omega_0^2}\delta\psi_0+\delta\phi_0-\frac{\omega_G^2}{\omega_0^2}\delta\phi_0\right)$$
$$= -i\frac{c}{B\omega_0}k_Zk_{\theta,0}\left(k_Z^2-k_{\theta,0}^2\right)\delta\phi_0\left(\delta\phi_Z-\frac{k_{\parallel,0}^2v_{\rm A}^2}{\omega_0^2}\delta\psi_Z\right), \quad (8)$$

with the term proportional to $\delta\phi_Z$ on the right hand side corresponding to RS contribution and $\delta\psi_Z$ term corresponding to MX contribution. The third term on the left hand side of Eq. (8) is the SAW continuum upshift due to geodesic curvature induced compression, and ω_G is the frequency of GAM [50].

Substituting Eq. (7) into Eq. (8), one then obtains the equation describing the modulation of RSAE by ZFZS,

$$k_{\perp,0}^2\mathscr{E}_0\delta\phi_0 = -i\frac{c}{B\omega_0}\left(k_Z^2-k_{\theta,0}^2-\frac{k_{\parallel,0}^2v_{\rm A}^2}{\omega_0^2}k_{\perp,0}^2\right)k_Zk_{\theta,0}\delta\phi_0\left(\delta\phi_Z-\alpha\delta\psi_Z\right) \quad (9)$$

with $\mathscr{E}_0$ being the RSAE dispersion relation, and $\alpha \equiv (k^2_{\parallel,0} v^2_{\rm A}/\omega^2_0)(-2k^2_{\theta,0})/(k^2_Z - k^2_{\theta,0} - k^2_{\parallel,0} v^2_{\rm A} k^2_{\perp,0}/\omega^2_0)$. Eq. (9) is general, and can be reduced to various limits depending on different plasma parameters, through the value of α dependence on $k_{\parallel,0}$, e.g., the mode dynamics described by Eq. (9) is similar to the TAE case as $(k^2_{\parallel,0} v^2_{\rm A}/\omega^2_0) \sim \mathcal{O}(1)$, and $|k_{\parallel,0}| \simeq 1/(2qR)$, and thus $\alpha \simeq 1$. On the other hand, the mode behavior gets close to a BAE with $k_{\parallel,0} \simeq 0$, and thus $\alpha \simeq 0$. For simplicity of investigation, the RSAE WKB dispersion relation can be adopt here as $\mathscr{E}_0 \simeq (1 - k^2_{\parallel,0} v^2_{\rm A}/\omega^2_0 - \omega^2_G/\omega^2_0)$, while the radial global dispersion relation [5] can be applied for more quantitative analysis.

Furthermore, subtracting Eq. (5) by $\alpha\times$Eq. (6), one can obtain,

$$\delta\phi_Z - \alpha\delta\psi_Z = -i\frac{c}{B}k_Z k_{\theta,0}\left[\frac{1 - k^2_{\parallel,0} v^2_{\rm A}/\omega^2_0}{\omega_Z \hat{\chi}_Z k_Z}(k_{r,0} - k_{r,0^*}) + \frac{\alpha}{\omega_0}\right]|\delta\phi_0|^2, \quad (10)$$

which can be substituted into Eq. (9), and obtain the general equation describing the self-modulation of RSAE, as

$$k^2_{\perp,0}\mathscr{E}_0\delta\phi_0 = -\left(\frac{c}{B}k_Z k_{\theta,0}\right)^2 \frac{1}{\omega_0}\left(k^2_Z - k^2_{\theta,0} - \frac{k^2_{\parallel 0} v^2_{\rm A}}{\omega^2_0}k^2_{\perp 0}\right)$$
$$\times \left[\frac{1 - k^2_{\parallel,0} v^2_{\rm A}/\omega^2_0}{\omega_Z \hat{\chi}_Z k_Z}(k_{r,0} - k_{r,0^*}) + \frac{\alpha}{\omega_0}\right]\delta\phi_0 |\delta\phi_0|^2. \quad (11)$$

Both ZF and ZC generation by RSAE are systematically accounted for in Eq. (11) on the same footing, with the first term in square brackets corresponds to the ZF generation while the second term corresponds to the ZC. On the one hand, which one is preferentially excited is shown in the Sec. 4. On the other hand, ZFZS generation can lead to RSAE saturation by scattering to linearly stable radial eigen-states. The nonlinear saturation level, can be determined by self-consistently solving Eq. (11) as a nonlinear Schrodinger equation [49], while a rough estimation is given in, e.g., Ref. [33], by separating the AE into a pump and its sidebands, and deriving the fixed point solution of the coupled nonlinear equations. Since RSAE linear properties is very sensitive to q-profile, the modulation of q-profile caused by the nonlinearly generated ZC may have a great impact on RSAE nonlinear saturation. This is discussed in the Sec. 5.

4. ZFZS spontaneous excitation by RSAE

To investigate the linear growth stage of the modulational instability, we follow the analysis of Ref. [30], and consider the fluctuation consists of a constant-amplitude pump wave $\Omega_P \equiv \Omega_P(\omega_P, \mathbf{k_P})$ and its upper and lower sidebands $\Omega_\pm \equiv \Omega_\pm(\omega_\pm, \mathbf{k}_\pm)$ due to the modulation of the ZFZS $\Omega_Z \equiv \Omega_Z(\omega_Z, \mathbf{k_Z})$ [37]. Here, subscripts 'P', '+' and '−' denote RSAE pump, upper and lower sidebands respectively, with $\delta\phi_0 = \delta\phi_P + \delta\phi_+$, $\delta\phi_{0^*} = \delta\phi_{P^*} + \delta\phi_-$, and

$$\delta\phi_P = A_P e^{i(n\phi - \hat{m}\theta - \omega_P t)}\sum_j e^{-ij\theta}\Phi_P(x - j) + c.c.,$$

$$\delta\phi_{\pm} = A_{\pm} e^{\pm i(n\phi - \hat{m}\theta - \omega_P t)} e^{i\left(\int \hat{k_Z} dr - \omega_Z t\right)} \sum_j e^{\mp ij\theta} \left\{ \begin{array}{c} \Phi_P (x-j) \\ \Phi_{P^*} (x-j) \end{array} \right\} + c.c..$$

Then the general nonlinear Eqs. (5)-(8) can be reduced to equations describing $\delta\phi_Z$, $\delta\phi_+$ and $\delta\phi_-$ generation by the fixed amplitude pump RSAE, while the feedback of ZFZS and RSAE sidebands to the pump wave is neglected, focusing on the initial stage of the nonlinear process. Considering the frequency/wave number matching condition ($\omega_{\pm} = \pm\omega_P + \omega_Z$, $\mathbf{k}_{\pm} = \pm\mathbf{k}_P + \mathbf{k}_Z$) imbedded in the above expressions, Eq. (5) can be reduced to

$$\omega_Z \hat{\chi}_Z \delta\phi_Z = -i\frac{c}{B} k_Z k_{\theta,P} \left(1 - \frac{k^2_{\parallel,P} v^2_{\mathrm{A}}}{\omega^2_P}\right) \left(\delta\phi_+ \delta\phi_{P^*} - \delta\phi_- \delta\phi_P\right), \tag{12}$$

with $(1 - k^2_{\parallel,P} v^2_{\mathrm{A}}/\omega^2_P)$ representing the competition of Reynolds and Maxwell stresses to break the pure Alfvénic state [21, 30]. On the other hand, Eq. (6) describing ZC excitation can be reduced to

$$\delta\psi_Z = -i\frac{c}{B} k_{\theta,P} k_Z \frac{1}{\omega_P} \left(\delta\phi_+ \delta\phi_{P^*} + \delta\phi_- \delta\phi_P\right). \tag{13}$$

The nonlinear equation describing RSAE sidebands generation through the ZFZS modulation to pump RSAE can be derived from Eq. (9) as

$$\begin{aligned} k^2_{\perp,\pm} \mathscr{E}_{\pm} \delta\phi_{\pm} = & -i\frac{c}{B\omega_{\pm}} \left(k^2_Z - k^2_{\theta,P} - \frac{k^2_{\parallel,P} v^2_{\mathrm{A}}}{\omega^2_P} k^2_{\perp,\pm}\right) \\ & \times k_Z k_{\theta,P} \left\{ \begin{array}{c} \delta\phi_P \\ \delta\phi_{P^*} \end{array} \right\} \left(\delta\phi_Z - \alpha\delta\psi_Z\right). \end{aligned} \tag{14}$$

Eqs. (12)-(14), are equivalent to Eqs. (34)-(36) for TAE cases as derived in Ref. [32], with the coefficient α generalized to include a broader parameter regime ($\alpha \simeq 1$ for TAE as discussed in Ref. [32]). Note that $k^2_{\perp,\pm} = k^2_{\perp,P} + k^2_Z$ and $k^2_{\perp,+}$ is used in the derivation later. Similarly, subtracting Eq. (12) by $\alpha\times$Eq. (13), one can obtain

$$\begin{aligned} \delta\phi_Z - \alpha\delta\psi_Z = i\frac{c}{B} k_Z k_{\theta,P} \Bigg[& \left(\frac{1 - k^2_{\parallel,P} v^2_{\mathrm{A}}/\omega^2_P}{\omega_Z \hat{\chi}_Z} + \frac{\alpha}{\omega_P}\right) \delta\phi_+ \delta\phi_{P^*} \\ & - \left(\frac{1 - k^2_{\parallel,P} v^2_{\mathrm{A}}/\omega^2_P}{\omega_Z \hat{\chi}_Z} - \frac{\alpha}{\omega_P}\right) \delta\phi_- \delta\phi_P \Bigg]. \end{aligned} \tag{15}$$

The modulational instability dispersion relation can then be derived by substituting $\delta\phi_{\pm}$ obtained from Eq. (14) into (15), as

$$\begin{aligned} 1 = -\hat{F} \left|\delta\phi_P\right|^2 \Bigg[& \left(\frac{1 - k^2_{\parallel,P} v^2_{\mathrm{A}}/\omega^2_P}{\omega_Z \hat{\chi}_Z} + \frac{\alpha}{\omega_P}\right) \frac{1}{\mathscr{E}_+} \\ & - \left(\frac{1 - k^2_{\parallel,P} v^2_{\mathrm{A}}/\omega^2_P}{\omega_Z \hat{\chi}_Z} - \frac{\alpha}{\omega_P}\right) \frac{1}{\mathscr{E}_-} \Bigg], \end{aligned} \tag{16}$$

with $\hat{F} = (ck_Z k_{\theta,P}/B)^2 (-k^2_Z + k^2_{\theta,P} + k^2_{\parallel,P} v^2_{\mathrm{A}} k^2_{\perp,+}/\omega^2_P)/(\omega_P k^2_{\perp,+})$ being a nonlinear coupling coefficient. Furthermore, considering RSAE sidebands still obey the dispersion relation of RSAE by $\mathscr{E}_\pm = \mathscr{E}_0(\omega_Z \pm \omega_P, k_Z)$, one can expand $\mathscr{E}_\pm$ along the RSAE characteristics, as $\mathscr{E}_\pm \simeq (\partial\mathscr{E}_0/\partial\omega_0)(\pm\omega_Z - \Delta)$ with $\Delta \equiv -k_Z^2(\partial^2\mathscr{E}_0/\partial k_r^2)/(2\partial\mathscr{E}_0/\partial\omega_0)$ being the frequency mismatch, describing the frequency shift of RSAE sidebands from the pump RSAE dispersion relation due to the ZFZS modulation. Denoting $\gamma \equiv -i\omega_Z$ and noting $\omega_\pm \simeq \pm\omega_P$, the modulational instability dispersion relation can be shown as

$$\gamma^2 = -\Delta^2 + \frac{2\hat{F}\,|\delta\phi_P|^2}{\partial\mathscr{E}_0/\partial\omega_0}\left(\frac{1 - k_{\parallel,P}^2 v_{\rm A}^2/\omega_0^2}{\hat{\chi}_Z} + \frac{\alpha}{\omega_P}\Delta\right). \tag{17}$$

Here, the first term on the right side of Eq. (17) is the threshold condition due to frequency mismatch and the second term represents the nonlinear drive. Thus, the ZFZS can be spontaneously excited when the nonlinear drive overcomes the threshold condition due to frequency mismatch, as the RSAE amplitude is large enough, or the nonlinear coupling is strong enough, as we address in the following discussion. Furthermore, the first term in brackets corresponds to ZF contribution to the nonlinear coupling, and the second term corresponds to ZC. For ZF contribution, there are two restrictions due to, first, the partial cancelation of RS and MX with the 'residual' drive due to deviation from ideal MHD limit due to plasma nonuniformity (reversed q-profile here) that breaks the pure Alfvénic state [21, 5], and second, the neoclassical shielding of ZF as shown by the $\hat{\chi}_Z$ in the denominator, with $\hat{\chi}_Z \sim q^2/\epsilon$, which is typically much larger than unity [40, 8].

For ZC contribution, there are also two important factors that crucially determine the nonlinear process, with the first being the sign of Δ, which is typically determined by specific plasma parameters. The ZC term is a driving term for $\Delta > 0$; while for $\Delta < 0$, the ZC term becomes a damping term for this nonlinear process; and then the condition for modulational instability becomes very stringent, with additional requirement for the ZF term being dominant over the ZC term, which is not easy to satisfy due to the two restrictions as addressed in the paragraph above. The other important factor is the value of α. As noted before, if RSAE localizes near the rational surface, $\alpha \simeq 0$ because of $k_\parallel \simeq 0$, meaning ZC generation is very weak and is similar to BAE case investigated in Ref. [41]; on the other hand, if RSAE localizes in the middle of two neighboring rational surfaces, $\alpha \simeq 1$, and this case is similar to the TAE case [30] with ZC generation preferred. So for RSAE, with $k_\parallel$ and its frequency determined by $q_{\rm min}$ and the corresponding mode numbers, both ZF and ZC generation could be dominant, depending on the specific plasma parameters, and should be investigated case by case.

5. Nonlinear RSAE saturation

In strongly nonlinear stage, the feedback of ZFZS and RSAE sidebands to the pump wave can not be neglected anymore, as the sideband amplitudes become comparable to that of the pump wave. It is indicated in Eq. (11) that ZFZS can play self-regulatory roles on RSAE nonlinear evolution by scattering RSAE into short radial wavelength stable domains, which may lead to RSAE saturation. The saturation level, can be derived from the coupled pump wave and sidebands equations, as is shown in, e.g., Ref. [51]. In addition, another related channel unique for RSAE nonlinear saturation may exist, due to the modulation to SAW continuum [28, 29] by the nonlinearly generated ZFZS (among which ZC playing a dominant role), considering the sensitive dependence of RSAE on SAW continuum accumulational point (for a visualization of RSAE physics dependence on $q_{\min}$, interested readers may refer to Ref. [14] and Fig. 3 therein). The nonlinear generated ZC is a toroidal current sharply localized around $q_{\min}$, which can generate a perturbed poloidal magnetic field and further modulate q-profile and thus the SAW continuum near $q_{\min}$. Thus, one can reasonable speculate that ZC plays an important role in RSAE saturation by modifying the equilibrium continuum, similar to the mechanism investigated in Ref. [28] and Ref. [29] for TAE.

Here, in consistency with the above speculation on the crucial role played by ZC, we consider a simplified case that ZC generation is dominant. This assumption is natural for scenarios with $(k_{\parallel,0}^2 v_{\rm A}^2/\omega_0^2) \sim \mathcal{O}(1)$, however, may also be important for other parameter regimes for the reasons addressed above. Thus, Eq. (11) can be simplified as

$$\mathscr{E}_0 \delta\phi_0 = 2\left(\frac{c}{B\omega_0} k_Z k_{\theta,0}\right)^2 \delta\phi_0 \left|\delta\phi_0\right|^2 . \tag{18}$$

Taking a two scale analysis by assuming $\omega_R = \omega_0 + i\partial_\tau$, and expanding $\mathscr{E}_0 \simeq (\partial\mathscr{E}_0/\partial\omega_0)(i\partial_\tau - i\gamma_R^L - \Delta)$ with Δ being the nonlinearity induced frequency shift and γ_R^L is the linear RSAE growth rate §, Eq. (18) becomes

$$\left[i\partial_\tau - i\gamma_R^L - \Delta - 2\left(\frac{c}{B\omega_0} k_Z k_{\theta,0}\right)^2 \frac{\left|\delta\phi_0\right|^2}{\partial\mathscr{E}_0/\partial\omega_0} \right] \delta\phi_0 = 0. \tag{19}$$

Eq. (19) describes the RSAE nonlinear evolution due to, scattering to different radial eigen-state (denoted by Δ) with different linear growth/damping rates (γ_R^L), and nonlinear self-modulation by ZC generation, and the saturation level can be estimated by balancing the frequency shift (Max $(|\gamma_R^L|, |\Delta|)$) and the nonlinear RSAE modulation by ZC, and one has

$$\left|\delta\phi_0\right|^2 = \frac{\partial^2 \mathscr{E}_0}{\partial k_r^2} \left(\frac{B\omega_0}{2ck_{\theta,0}}\right)^2 . \tag{20}$$

Eq. (20) describes the RSAE saturation level due to scattering by self-generated ZC to neighboring (linearly more stable) radial eigen-states, assuming $\Delta \gg \gamma_R^L$. Cases for $\Delta \ll \gamma_R^L$ can be evaluated similarly. The RSAE saturation level can be estimated by substituting typical parameters into the expression. Furthermore, Eq. (20) can be substituted into Eq. (6), and the saturation level of the perturbed poloidal magnetic field δB_θ can be estimated as

$$\delta B_\theta \sim \frac{B_0 k_{\parallel,0} k_Z^2}{4k_{\theta,0}} \left.\frac{\partial^2 \mathscr{E}_0}{\partial k_r^2}\right|_{k_r=0} , \tag{21}$$

§ Note the linear RSAE growth/damping rate here, which is not included in Eq. (17) assuming RSAE sidebands damping rates are typically smaller than frequency mismatch.

resulting in a modulation of local q_{min} by

$$\delta q \sim \; - q_{\rm min}\frac{B_0 k_{\parallel,0}k_Z^2}{4B_\theta k_{\theta,0}}\left.\frac{\partial^2\mathscr{E}_0}{\partial k_r^2}\right|_{k_r=0}. \tag{22}$$

In deriving Eqs. (21) and (22), we have noted $\delta\psi = \omega\delta A_\parallel/(ck_\parallel)$ and $\delta\mathbf{B} = \nabla\times\delta A_\parallel\mathbf{b}$. Typical parameters can be adopted, i.e., $q_{\rm min}\sim\mathcal{O}(1)$, $B_0/B_\theta\sim qR/a$, $k_\parallel\sim 1/(qR)$, $k_{\theta,0}\rho_d\sim\mathcal{O}(1)$ with ρ_d being the drift orbit radius and

$$k_Z^2\left.\frac{\partial^2\mathscr{E}_0}{\partial k_r^2}\right|_{k_r=0}\sim\frac{4\Delta}{\omega}, \tag{23}$$

which can be reasonably assumed as $\sim\mathcal{O}(0.1)$. Thus, one can roughly estimate that $\delta q/q\sim\mathcal{O}(10^{-3})$. Noting that the modification to local RSAE continuum frequency is $\sim O(n\delta q v_{\rm A}/(qR))$, and that for reactor burning plasma with $\rho_d/a\sim O(10^{-2})$, most unstable RSAEs are characterized by $n\gtrsim O(10)$ [9], the modification to local SAW continuum is comparable to the RSAE linear growth rate γ_R^L or frequency differences between different radial eigen-states ($\sim\Delta$). Thus, one expects that the ZC induced SAW continuum modification in the vicinity of $q_{\rm min}$, plays an important role in RSAE nonlinear saturation, though the self-consistent study analogous to Refs. [28, 29] is not presented. The systematic investigation of RSAE saturation due to nonlinear modification of SAW continuum and the resulting enhanced continuum damping, will be presented in a separate publication.

6. Conclusion and Discussion

The general equations describing RSAE self-modulation via nonlinear excitation of ZFZS are derived using gyrokinetic theory, which is then applied to study the spontaneous ZFZS excitation via modulational instability as well as RSAE nonlinear saturation due to scattering to more stable radial eigen-states. It is found that, both ZF and ZC can be dominant in the spontaneous excitation by RSAE, depending on the specific plasma parameters, especially $q_{\rm min}$ that determines the RSAE parallel wavenumber and frequency. The obtained general modulational instability dispersion relation for ZFZS excitation by RSAE, Eq. (17), can recover the results of TAE [30] and BAE [41] in the proper limits, i.e., by taking $k_\parallel v_{\rm A}/\omega\rightarrow 1$ and 0, respectively. The properties of ZFZS generation by RSAE, noting that the typical RSAE parallel wavenumber and frequency are in between those of TAE and BAE, can be understood based on the knowledge obtained from TAE [30] and BAE [41].

An interesting step forward is, the saturation level of RSAE is qualitatively estimated by balancing the nonlinear scattering by ZFZS with the frequency differences between different radial eigen-states ($\sim\Delta$), assuming ZC playing a dominant role in the RSAE scattering. The corresponding ZC saturation level as well as the modification to local q_{min}, are also estimated. It is found that, the resulting modification to local SAW continuum accumulational point frequency, can be at least, comparable to the RSAE linear growth rate or frequency mismatch between different radial eigen-states for burning plasma scenarios with most unstable RSAEs characterized by $n \gtrsim O(10)$ [9]. Thus, the modification of local SAW continuum by ZC is expected to contribute significantly to RSAE saturation [28, 29].

The above estimation based on Eq. (18), assumes dominant ZC generation by taking $|k_\parallel v_A/\omega| \sim 1$, is valid for other parameter regimes due to the weak coupling coefficients of ZF generation, except for the cases that $q_{\rm min}$ is localized very close to a low-order rational surface, such that the RSAE properties is close to BAE with predominantly ZF generation [41]. The logic underlying the reasoning presented in Sec. 5 is that, the RSAE and ZFZS saturation level are estimated without accounting for the modification to SAW continuum, which is then used to estimate the modification to SAW continuum by the saturated ZC, and found that the modification to SAW continuum could be comparable or even more important than the ZFZS shearing. Thus, the obtained results indicate that, the modification to SAW continuum, will start to influence the RSAE nonlinear evolution, before it saturates due to self-modulation via ZFZS generation. Our work thus indicates that multiple processes may contribute comparably to the RSAE saturation, and should be accounted for on the same footing, based on the solid understanding of each individual process, to properly assess the saturation and thus EP transport by RSAE. This is of particular importance since RSAEs are expected to be strongly excited by core-localized fusion alphas in future reactors characterized by the advanced reversed shear scenarios.

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Data Availability

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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