Darwin Model in Plasma Physics Revisited

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Dispersion relations from the Darwin (a.k.a., magnetoinductive or magnetostatic) model are given and compared with those of the full electromagnetic model. Analytical and numerical solutions show that the errors from the Darwin approximation can be large even for low-frequency waves with phase velocity close to or larger than the speed of light. Besides missing two wave branches associated mainly with the electron dynamics, in the Darwin model the coupling branch of the electrons and ions is modified to become a new artificial branch that incorrectly represents the coupling dynamics of the electrons and ions.

I. INTRODUCTION

The Darwin (a.k.a., magnetoinductive or magnetostatic) model, which originated from a second order Lagrangian in terms of v/c (v is the velocity of a charged particle and c is the speed of light) for charged particle motion [2], is widely used for low-frequency (nonradiative limit) plasma simulations. Application of this model to plasma simulations was theoretically discussed by Kaufman and Rostler[6]. The widely used particle-in-cell (PIC) simulation in the framework of this model for both 1D and 2D was developed in Refs.[1, 9] and can be found in the review papers [3, 5] or textbook [13]. New analytical investigations and applications of the Darwin model can be found in, e.g., Refs.[7, 8]. Vlasov-Darwin codes have also been proposed more recently (see e.g.,[10]).

It is known that the high-frequency electromagnetic radiations are neglected in Darwin approximation. This is also why Darwin model is useful for simulating low-frequency phenomena, because that the time step needs not to be small to resolve the high-frequency electromagnetic waves. However, that how this approximation affects the low-frequency waves has not been understood clearly. Most existing papers focused on the simulation aspects. Few carefully checked the *quantitative* differences of the results between the Darwin approximation and the full electromagnetic (EM) model. Busnardo-Neto *et al* [1] used a simplified dispersion relation [4] under the cold plasma assumption to benchmark. The properties of the Darwin model dispersion relations were discussed in Refs.[6, 13].

In this work, we give the limits of validity of the Darwin model by carefully comparing the dispersion relations from the Darwin model and the full EM model.

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II. BASIC EQUATIONS

All field variables are divided into two parts in the Darwin model [1]: the transverse (T, divergence free) part and the longitudinal (L, curl free) part.

$$\begin{aligned} \boldsymbol{E} &= \boldsymbol{E}_L + \boldsymbol{E}_T, \nabla\cdot\boldsymbol{E}_T = 0, \nabla\times\boldsymbol{E}_L = 0, \\ \boldsymbol{B} &= \boldsymbol{B}_T, \nabla\cdot\boldsymbol{B}_T = 0, \\ \boldsymbol{J} &= \boldsymbol{J}_L + \boldsymbol{J}_T, \nabla\cdot\boldsymbol{J}_T = 0, \nabla\times\boldsymbol{J}_L = 0. \end{aligned} \tag{1}$$

The Maxwellian equations are,

$$\begin{aligned} \nabla\cdot\boldsymbol{E}_L &= \rho/\varepsilon_0, \\ \nabla\cdot\boldsymbol{B} &= 0, \\ \nabla\times\boldsymbol{E}_T &= -\partial\boldsymbol{B}/\partial t \text{ (or, } \nabla\times\boldsymbol{E} = -\partial\boldsymbol{B}/\partial t), \\ \nabla\times\boldsymbol{B} &= \mu_0\boldsymbol{J} + \mu_0\varepsilon_0\partial\boldsymbol{E}_L/\partial t + \underbrace{\mu_0\varepsilon_0\partial\boldsymbol{E}_T/\partial t}_{\text{dropped in Darwin model}} \end{aligned} \tag{2}$$

For the PIC simulation, the governing equations are as follows

$$\begin{aligned} \nabla\cdot\boldsymbol{E}_L &= \rho/\varepsilon_0, \\ \nabla^2\boldsymbol{B} &= -\mu_0\nabla\times\boldsymbol{J}, \\ \nabla^2\boldsymbol{E}_T &= \mu_0\partial\boldsymbol{J}/\partial t + \mu_0\varepsilon_0\partial^2\boldsymbol{E}_L/\partial t^2. \end{aligned} \tag{3}$$

III. COLD PLASMA DISPERSION RELATION

For simplicity, we study the cold plasma dispersion relation firstly. Without loss of generality, we assume the external magnetic field $\boldsymbol{B_0} = (0, 0, B_0)$ and the wavevector $\boldsymbol{k} = (k\sin\theta, 0, k\cos\theta)$ and obtain

$$D = \begin{pmatrix} S - n'^2\cos^2\theta & -iD & n'^2\sin\theta\cos\theta \\ iD & S - n'^2 & 0 \\ n'^2\sin\theta\cos\theta & 0 & P - n'^2\sin^2\theta \end{pmatrix}, \tag{4}$$

where [11]

$$S = 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2 - \omega_{cs}^2}, \tag{5}$$

$$D = \sum_s \frac{\omega_{cs}\omega_{ps}^2}{\omega(\omega^2 - \omega_{cs}^2)}, \tag{6}$$

$$P = 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2}, \tag{7}$$

$\omega_{cs} = e_s B_0/m_s$ ($e_e = -e$), $\omega_{ps}^2 = n_{s0}e_s^2/\varepsilon_0 m_s$. The dispersion relation is then

$$\det[\mathrm{D}(\omega, \mathrm{k})] = 0. \tag{8}$$

The index of the refraction is $\boldsymbol{n} = c\boldsymbol{k}/\omega$, with $c = 1/\sqrt{\mu_0\varepsilon_0}$ the speed of light. In Eq.(4), for the traditional EM model, $n'^2 = n^2$. For the Darwin model, it becomes $n'^2 = n^2 + 1$. This replacement happens not only for high-frequency waves but also for low-frequency waves.

With an assumption of a single ion species ($s = e, i$), Eq.(8) for traditional EM model can be written to (see also Ref.[12])

$$c_{10}\omega^{10} - c_8\omega^8 + c_6\omega^6 - c_4\omega^4 + c_2\omega^2 - c_0 = 0, \tag{9}$$

where

$$\begin{aligned}
c_0 &= c^4k^4\omega_{ce}^4\omega_{ci}^4\omega_p^2\cos^2\theta,\\
c_2 &= c^4k^4\\
&\quad [\omega_p^2(\omega_{ce}^2 + \omega_{ci}^2 - \omega_{ci}\omega_{ce})\cos^2\theta + \omega_{ci}\omega_{ce}(\omega_p^2 + \omega_{ci}\omega_{ce})]\\
&\quad +c^2k^2\omega_p^2\omega_{ci}\omega_{ce}(\omega_p^2 + \omega_{ci}\omega_{ce})(1 + cos^2\theta),\\
c_4 &= c^4k^4(\omega_{ce}^2 + \omega_{ci}^2 + \omega_p^2) + 2c^2k^2(\omega_p^2 + \omega_{ci}\omega_{ce})^2\\
&\quad +c^2k^2\omega_p^2(\omega_{ce}^2 + \omega_{ci}^2 - \omega_{ci}\omega_{ce})(1 + cos^2\theta)\\
&\quad +\omega_p^2(\omega_p^2 + \omega_{ci}\omega_{ce})^2,\\
c_6 &= c^4k^4 + (2c^2k^2 + \omega_p^2)(\omega_{ce}^2 + \omega_{ci}^2 + 2\omega_p^2)\\
&\quad +(\omega_p^2 + \omega_{ci}\omega_{ce})^2,\\
c_8 &= \left(2c^2k^2 + \omega_{ce}^2 + \omega_{ci}^2 + 3\omega_p^2\right),\\
c_{10} &= 1.
\end{aligned} \tag{10}$$

Here, $\omega_{ce} = |\omega_{ce}|$ and $\omega_p^2 = \omega_{pe}^2 + \omega_{pi}^2$. Eq.(9) is a fifth order equation for ω^2, which has five branches of solutions. This equation has also been verified by a general magnetized multi-fluid plasma dispersion relation solver PDRF[14].

For the Darwin model, the dispersion relation is

$$c_6\omega^6 - c_4\omega^4 + c_2\omega^2 - c_0 = 0, \tag{11}$$

where

$$\begin{aligned}
c_0 &= c^4k^4\omega_{ce}^4\omega_{ci}^4\omega_p^2\cos^2\theta,\\
c_2 &= c^4k^4[\omega_{ci}^2\omega_p^2 + \omega_{ce}^2\omega_{ci}^2 + \omega_{ce}^2\omega_p^2\\
&\quad -\omega_p^2\left(\omega_{ce}^2 + \omega_{ci}^2 - \omega_{ce}\omega_{ci}\right)\sin^2\theta]\\
&\quad +c^2k^2\omega_p^2\omega_{ce}\omega_{ci}\left[\omega_{ce}\omega_{ci}\sin^2\theta + \omega_p^2(1 + \cos^2\theta)\right],\\
c_4 &= c^4k^4\left(\omega_{ce}^2 + \omega_{ci}^2 + \omega_p^2\right)\\
&\quad +c^2k^2\omega_p^2\left[2\omega_p^2 + 2\omega_{ce}\omega_{ci} + \left(\omega_{ce}^2 + \omega_{ci}^2 - \omega_{ce}\omega_{ci}\right)\sin^2\theta\right]\\
&\quad +\omega_p^4\left[\omega_{ce}\omega_{ci}\sin^2\theta + \omega_p^2\right],\\
c_6 &= \left(c^2k^2 + \omega_p^2\right)^2.
\end{aligned} \tag{12}$$

Eq.(11) is a cubic equation for ω^2, which only has three branches of solutions. It is easily shown that the eliminated two branches of waves have a higher phase velocity which is close to the speed of light when k is large ($\omega^2 \sim k^2c^2$, for a large wavevector k).

IV. ANALYTICAL INVESTIGATIONS

Eqs.(9) and (11) are used to examine the errors of the Darwin approximation with arbitrary parameters of k, θ, ω_{pe}, ω_{ce} and m_i/m_e.

For $n \simeq 1$ or $n < 1$, the relative difference between $n'^2_{\mathrm{Darwin}} = n^2 + 1$ and $n'^2_{\mathrm{EM}} = n^2$ would be large. It should be noted that even for some *low-frequency* branches, the Darwin approximation can also be invalid because the phase velocity of the low-frequency modes can be close to or larger than the speed of light for some middle or small k (See detailed numerical results in the following sections).

For the case with a small k ($k \to 0$) or a very long wave length, the solution for Darwin model is

$$\omega^2 = \omega_{ce}\omega_{ci}\sin^2\theta + \omega_p^2, \tag{13}$$

which is dependent on θ, whereas the solutions for EM model are given by

$$\begin{aligned}
&\omega^6 - (\omega_{ce}^2 + \omega_{ci}^2 + 3\omega_p^2)\omega^4 +\\
&[\omega_p^2(\omega_{ce}^2 + \omega_{ci}^2 + 2\omega_p^2) + (\omega_p^2 + \omega_{ci}\omega_{ce})^2]\omega^2 -\\
&\omega_p^2(\omega_p^2 + \omega_{ci}\omega_{ce})^2 = 0,
\end{aligned} \tag{14}$$

which is independent on θ, and yields three branches of solutions: $\omega^2 = \omega_p^2$ and $\omega^2 = \{(\omega_{ce}^2 + \omega_{ci}^2 + 2\omega_p^2) \pm \sqrt{[(\omega_{ce} + \omega_{ci})^2 + 4\omega_p^2](\omega_{ce} - \omega_{ci})^2}\}/2$. Eq.(13) is unreasonable because θ is meaningless when $k = 0$. This artificial solution is the same as the EM solution only when $\theta = 0$. These results indicate that in the Darwin model, not only are two higher frequency branches associated the electron dynamics missing, but also the electron-ion coupling mode with a long wavelength is falsely represented by the new artificial branch. It is suggested that the electron response to an external long wavelength perturbation is misinterpreted in the Darwin model.

For the case with $k \to \infty$, the solutions should be the same for both models due to $n'^2_{\mathrm{Darwin}} \simeq n'^2_{\mathrm{EM}} = n^2 \gg 1$ [11],

$$\tan^2\theta = -\frac{P}{S}. \tag{15}$$

Eq.(15) indicates that for $k \to \infty$, the three solutions of ω^2 for both models will approach to the same three constant-frequencies (see Fig.1).

For a system with an Alfvén wave being dominant (following section 2.4 in Ref.[11]), we have

$$\begin{aligned}
\omega^2 &= k^2v_A^2\\
&= k^2\frac{c^2}{\left(1 + \frac{\omega_{pi}^2}{\omega_{ci}^2}\right)\frac{(1+\cos^2\theta)\pm(1-\cos^2\theta)}{2\cos^2\theta}\underbrace{-1}_{\text{for Darwin model}}}
\end{aligned} \tag{16}$$

The '-1' term in Eq.(16) is resulted from the difference of the EM model and the Darwin model. It's easily shown that when $v_A/c \ll 1$, the difference is small. Thus, the Darwin model is a good approximation for studying the Alfvén wave dominant process such as magnetic reconnection.

Parameters (m_i/m_e, kc/ω_{ce}, ω_{pe}/ω_{ce}, θ)	EM model	Darwin model
(1836, 1.0, 0.833, 0)	5.4446E-4, **0.5144**, 0.8336, (1.1727, 1.6583)	5.4446E-4, **0.5903**, 0.8336
(1836, 10.0, 0.833, 0)	5.4466E-4, 0.8336, 0.9930, (10.0315, 10.0385)	5.4466E-4, 0.8336, 0.9931
(1836, 0.1, 0.833, 0)	5.2549E-4, 0.0144, (0.4880,) 0.8336, (1.4735)	5.2549E-4, 0.0147, 0.8336
(1836, 1.0, 0.833, 45)	5.4421E-4, 0.3273, **0.8983**, (1.2314, 1.6291)	5.4421E-4, 0.3399, **1.0240**
(1836, 1.0, 0.833, 90)	1.0973E-18, 0.0132, **0.9281**, (1.3019, 1.5900)	1.0973E-18, 0.0132, **1.1335**
(4, 1.0, 0.833, 0)	0.2153, **0.5427**, 0.9317, (1.2666, 1.6892)	0.2165, **0.6180**, 0.9317
(4, 1.0, 0.833, 45)	0.1865, **0.4072**, 0.9974, (1.3093, 1.6604)	0.1875, **0.4280**, 1.0985
(4, 1.0, 0.833, 90)	1.9738E-17, 0.3152, **1.0343**, (1.3668, 1.6215)	1.9738E-17, 0.3213, **1.2041**

TABLE I: $\Re(\omega) \geq 0$ solutions of the dispersion relation for the EM and Darwin model. The results show that the Darwin approximation can be invalid at least for some low frequencies modes.

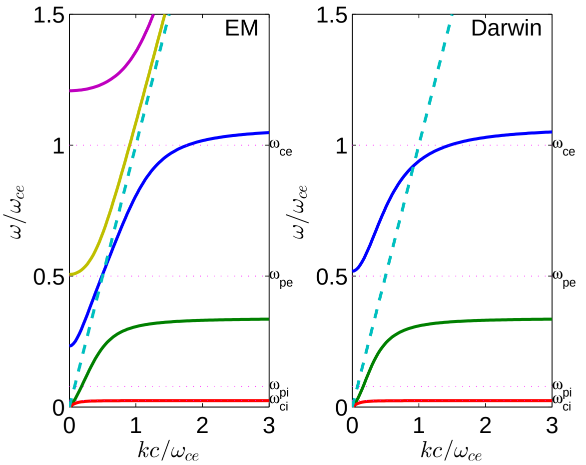

FIG. 1: (Color online) ω vs. k, with $m_i/m_e = 40$, $\omega_{pe}/\omega_{ce} = 0.5$ and $\theta = 45°$, where the green dashed line is referred to $\omega = kc$.

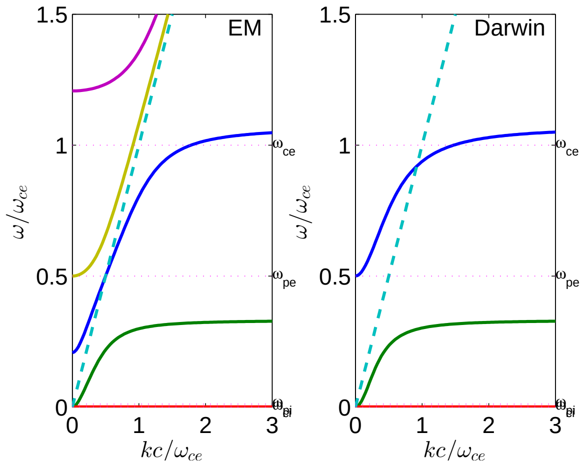

FIG. 2: (Color online) ω vs. k, with $m_i/m_e = 1836$. Other parameters are the same as in Fig.1.

V. NUMERICAL SOLUTIONS

In this section, Eqs.(9) and (11) are numerically solved to examine the differences between the EM and Darwin solutions.

To quantitatively examine differences of two models, we solve the dispersion relation under several typical parameters in Table I, where the extra two solutions in the EM model are labeled in the brackets. It is found from Table I that: 1. All roots are real numbers, which is reasonable, because there is no free energy in the system. Sturm's theorem can be used to show all solutions ω^2 for Eqs.(9) and (11) are positive real numbers; 2. The differences between two models are small for large k ($kc/\omega_{ce} = 10$) and small k ($kc/\omega_{ce} = 0.1$); 3. However, several roots (e.g., the solutions with **bold face** in the table) exist apparently different when $kc/\omega_{ce} \simeq 1$, which suggests that the Darwin approximation may be invalid at least for some middle k.

Fig.1 shows a typical result of ω vs. k, with $m_i/m_e = 40$, $\omega_{pe}/\omega_{ce} = 0.5$ and $\theta = 45°$. It is clearly indicated that the solutions of the two lowest frequency branches (around ω_{pi} and ω_{ci}) are nearly identical between the EM and Darwin model, whereas the third (around ω_{ce}) branch in the Darwin model is close to the fourth branch (around ω_{pe}) of the EM model when $kc/\omega_{ce} \ll 1$ but still the same as the third branch (around ω_{ce}) of EM model when $kc/\omega_{ce} \gg 1$. When $kc/\omega_{ce} < 1$, the third branch solution of Darwin model differs apparently from the solution of the EM model, which further indicates that the third branch in Darwin model would not truly describe the electron dynamics as discussed in Sec.IV. It should be noted that the reason to choose the low mass ratio of m_i/m_e is in order to increase the visibility of the ion branches of the waves fitting in one figure and the qualitative results will not change with a large mass ratio (Fig.2).

For the case with a middle wave vector k ($kc/\omega_{ce} = 1$), the dependence between ω and θ is shown in Fig. 3. The third branch (around ω_{ce} and ω_{pe}) of the Darwin model does not represent anyone of the modes in the EM model.

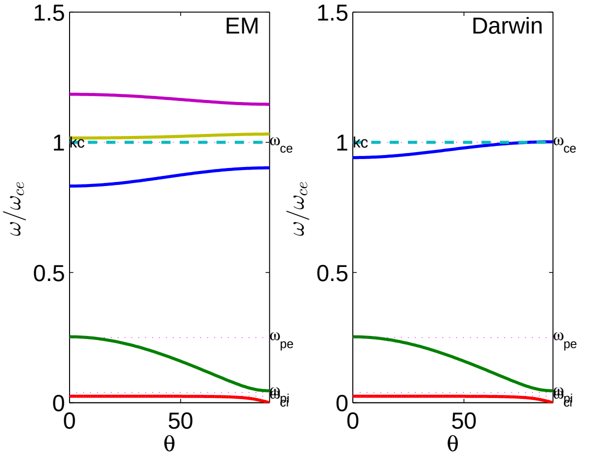

FIG. 3: (Color online) ω vs. θ, with $m_i/m_e = 40$, $\omega_{pe}/\omega_{ce} = 0.5$ and $k = 1.0\omega_{ce}/c$.

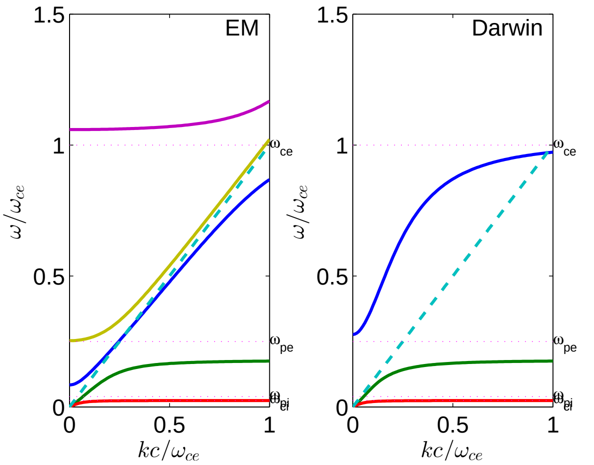

FIG. 4: (Color online) ω vs. k, with $\omega_{pe}/\omega_{ce} = 0.25$. Other parameters are the same as in Fig.1.

The third branch in the Darwin mode depends on the parameters (m_i/m_e, kc/ω_{ce}, ω_{pe}/ω_{ce}, θ). The deviation of the results from the Darwin and EM models for a small ω_{pe}/ω_{ce} becomes much larger as shown in Fig.4. By examining the parameter regimes with larger differences between the Darwin and EM models in Figs.1-4, it can be concluded that the Darwin approximation is invalid when the phase velocity of the low frequency modes is close to or larger than the speed of light, which is consistent with the analytical investigations in Sec.IV.

VI. KINETIC DISPERSION RELATIONS FOR HOT PLASMAS

The kinetic approach will bring Landau damping and modify the dispersion relation. As done in the cold plasma, we need to make a similar modification of the

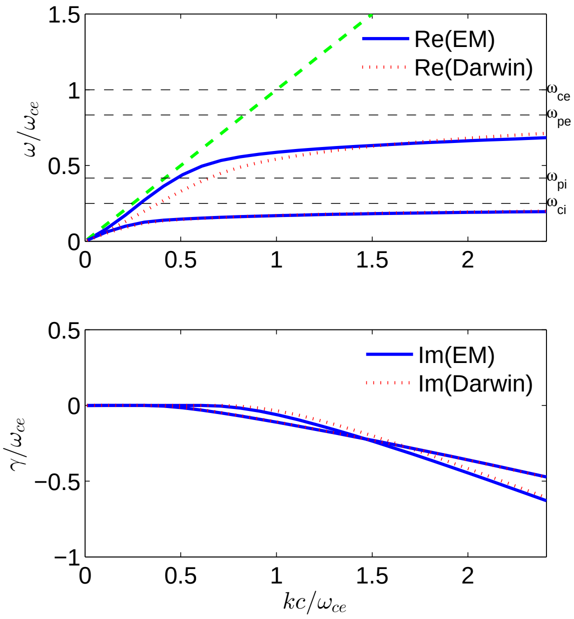

FIG. 5: (Color online) Kinetic solutions with parallel propagation for the Darwin and EM models, with $m_i/m_e = 4$, $\omega_{\rm ce}/\omega_{\rm pe} = 1.2$, $c/v_{\rm the} = 5$ and $T_i/T_e = 1$. The green dashed line is referred to $\omega = kc$.

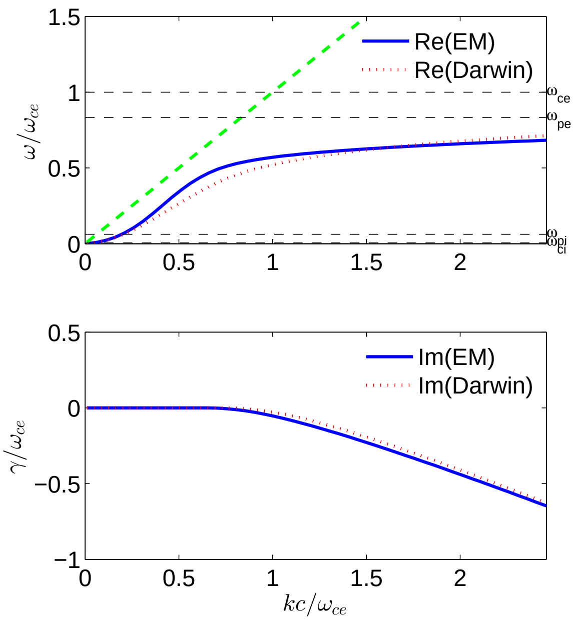

FIG. 6: (Color online) Kinetic solutions with parallel propagation for the Darwin and EM models, with $m_i/m_e = 1836$. Other parameters are the same as in Fig.5. Only the electron branch is shown.

well-known hot plasma dispersion relation matrix [11],

$$\begin{vmatrix} \epsilon_{xx} - n'^2 \cos^2\theta & \epsilon_{xy} & \epsilon_{xz} + n'^2 \sin\theta\cos\theta \\ \epsilon_{yx} & \epsilon_{yy} - n'^2 & \epsilon_{yz} \\ \epsilon_{zx} + n'^2 \sin\theta\cos\theta & \epsilon_{zy} & \epsilon_{zz} - n'^2 \sin^2\theta \end{vmatrix} = 0, \tag{17}$$

i.e., n'^2 explicitly included in the five terms must be replaced by $n'^2 = n^2$ in the EM model and $n'^2 = n^2 + 1$ in the Darwin model as in Eq.(4) while all other terms implicitly including n' remain unchanged.

Because it is impossible to solve the general kinetic dispersion relation analytically, we consider a parallel propagation case only. With a Mawellian equilibrium distribution, the dispersion relation becomes (note: $\omega_{ce} < 0$)

$$D(k,\omega) = 1 - n'^2 + \sum\nolimits_s \frac{\omega_{ps}^2}{\omega\sqrt{2}kv_{ths}} Z\left(\frac{\omega \pm \omega_{cs}}{\sqrt{2}kv_{ths}}\right) = 0, \tag{18}$$

where $n'^2 = n^2$ for the full EM model and $n'^2 = n^2 + 1$ for the Darwin model. Eq.(18) are also used in Ref. [1] for the benchmark.

Fig.5 shows a typical solution where a large difference between the EM model and Darwin approximation model is around $kc/\omega_{ce} = 0.5$. To show $\omega \simeq \omega_{ci}$ branch clearly, we have artificially chosen a small m_i/m_e. Actually, with the real mass ratio $m_i/m_e = 1836$, the differences of the two models are quite the same as shown in Fig.6. It seems that the Darwin model affects little to the Landau damping rates as shown in Figs.5 and 6.

VII. SUMMARY

Based on the analytical and numerical analysis of the cold and hot plasma dispersion relations from the Darwin and EM models, we conclude that: (1) The electron and ion coupling modes in the Darwin mode falsely describe the electron responses of a long wave perturbation because this branch of the dispersion relations has a rather different property as that in the EM model in the regime with a long wavelength. (2) The high-frequency electromagnetic waves mainly associated with the electron dynamic at small k are eliminated totally. (3) The low-frequency modes related with the ion dynamics in the Darwin model are in an excellent agreement with that in the EM model.

Generally, the Darwin approximation would be invalid when the phase velocity of a low-frequency wave is close to or larger than the speed of light.

Acknowledgments

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