Collapse and condensation with feedback as modeled by a generalized nonlinear Schrödinger equation

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Abstract

The effect of self-driven feedback control on post-collapse turbulent condensation [Cui et al., Phys. Rev. E **87**, 053104 (2013)] in a nonlinear dispersive and dissipative system governed by a two-dimensional generalized nonlinear Schrödinger equation is investigated. The evolution of an isolated initial pulse is followed numerically. It is found that with the feedback control the wavelength of the asymptotic turbulent condensate can to a degree be controlled.

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1. Introduction

The nonlinear Schrödinger equation (NSE) has been widely used for studying the evolution of systems involving wave or other dynamics and their nonlinear interactions [1-9]. The classical NSE describes the evolution of the envelope of a wave or wave-like field, including the effects of group velocity dispersion and ponderomotive response of the medium. When generalized to include diffusion and nonlinear damping/growth effects, the coefficients of the group-velocity dispersion and nonlinear terms in the NSE become complex. The equation is then also in the form of a generalized nonlinear reaction-diffusion equation [10-15], which likewise has a wide range of applications. The generalized NSE (GNSE) is therefore a useful model for studying nonequilibrium phenomena in nonlinear dissipative open systems, which often exhibit interesting local as well as global self-organized regular behavior coexisting with a nonthermal or turbulent background [16].

From the evolution of a conservation integral of the classical NSE, Zakharov [1] argued that spatial collapse of modulationally unstable Langmuir waves can occur [1,6-8,10-12]. To follow mathematically the evolution of highly singular phenomena such as collapse, in particular what happens afterwards, in a conservative system is generally difficult. Recently it has been shown that a two-dimensional (2D) GNSE can model the collapse of an initially localized perturbation into a turbulent state consisting of large amplitude fluctuations of the shortest (as allowed by the interaction characteristics or numerical accuracy) wavelengths [10-12]. The inverse mode cascade and/or decay following the collapse can then lead to turbulent asymptotic states with wide as well as narrow (e.g., nearly single-mode) spectra, corresponding to strong turbulence and turbulent condensation, respectively. However, it is difficult to preset or control the asymptotic state of the process, in particular, the wavelength of the condensed mode in the case of turbulent condensation. In this paper, we show that with sufficient self-driven feedback control one can realize post-collapse condensation to the longest wavelength modes allowed by the system.

2. GNSE with nonlinear feedback control

For convenience of comparison, the formulation and parameters here follow closely that of Ref. 12, which considered collapse and turbulent condensation without the feedback control. The GNSE including a simple feedback term can be written as

$$iE_t + p\nabla^2 E + \left(V(x,y) + q|E|^2 + \beta E^*\right)E = 0\,, \qquad (1)$$

where $p = p_r + ip_i$ and $q = q_r + iq_i$ are complex, and V and β are real. The physical meanings of the terms in (1) depend on the system being considered [1-9]. For definitiveness, we shall use the terminology of nonlinear wave dynamics, so that $E(t,x,y)$ represents the envelope of the wave field, $p_r\nabla^2 E$ the group velocity dispersion, $p_i\nabla^2 E$ the effect of viscosity, V the effect of an external field or inhomogeneity of the medium, $q_r|E|^2 E$ the (ponderomotive) nonlinear response [1,3,7,8] of the latter, $q_i|E|^2 E$ a magnitude-dependent nonlinear damping, and $\beta E^* E$ $\left(=\beta|E|^2\right)$ the feedback control. In particular, for stimulated light scattering in optical fiber, the feedback control term would correspond to cleaving one end of the long fiber at $90°$ [17-19]. The self-regulated feedback control of the incident light is then realized through the nonlinear interaction of the reflected light (which is out of phase) with the incident light. The external potential is [20]

$$V(x,y) = V_0\left\{1-\operatorname{sech}^2\left[\left(x^2+y^2\right)/a^2\right]\right\}, \tag{2}$$

and the initial pulse is given by

$$E_0(x,y) = E_0\exp\left[-\left(x^2+y^2\right)/c^2\right], \tag{3}$$

which is spatially narrower than the external potential even if a and c are of the same order, as shown in Fig. 1 for $V_0=6$, $E_0=0.1$, $a=4$, and $c=1.5$.

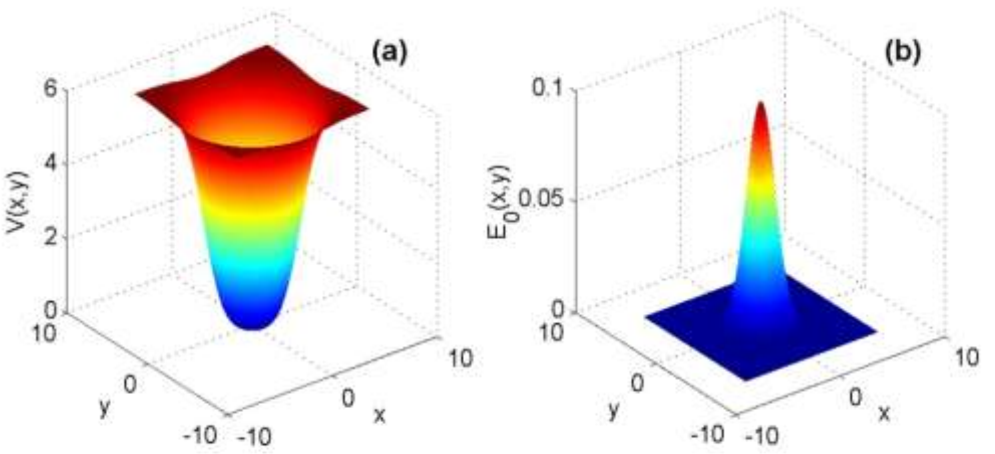

Fig. 1 (Color online.) The external potential $V(x,y)$ and the small initial perturbation $E_0(x,y)$.

One can obtain from (1) for the evolution of the total system “energy” the relation

$$\partial_t \int |E|^2 dxdy = 2\int \left[p_i |\nabla E|^2 - q_i |E|^4 + \beta E_i |E|^2 \right] dxdy, \tag{4}$$

which shows that positive values of p_i and q_i correspond to gain and loss, respectively, of the total energy.

We are interested in systems whose total energy can become stationary during the evolution, such as for $p_i > 0$ and $q_i > 0$. In this case the right-hand side of (4) can vanish during the evolution, so that the total energy becomes constant. This occurs if the summed effect of viscous growth and nonlinear damping of all the modes in the system become exactly balanced. The system as a whole then appears to be adiabatic even though there is local energy input and dissipation. However, with finite β, it is unlikely that the right hand side of (4) can vanish identically. Here we are interested in how does the feedback affect the post-collapse evolution.

3. **Numerical results**

We solve (1) numerically for $p = 0.5 + 0.05i$ and $q = 0.6 + 0.5i$ [12], and the $V(x, y)$ and $E_0(x, y)$ as given in (2) and (3) and shown in Fig. 1. That is, we consider a system with positive group dispersion ($p_r > 0$), viscous heating or growth ($p_i > 0$), modulational frequency up-shift ($q_r > 0$), and magnitude-dependent damping ($q_i > 0$), in the presence of a symmetric external potential $V(x, y)$. For the numerical solution, the split-spectrum-Runge-Kutta method [13-15,21] is used for the space and time evolutions. The simulation box has 256×256 grids. Periodic boundary conditions are used.

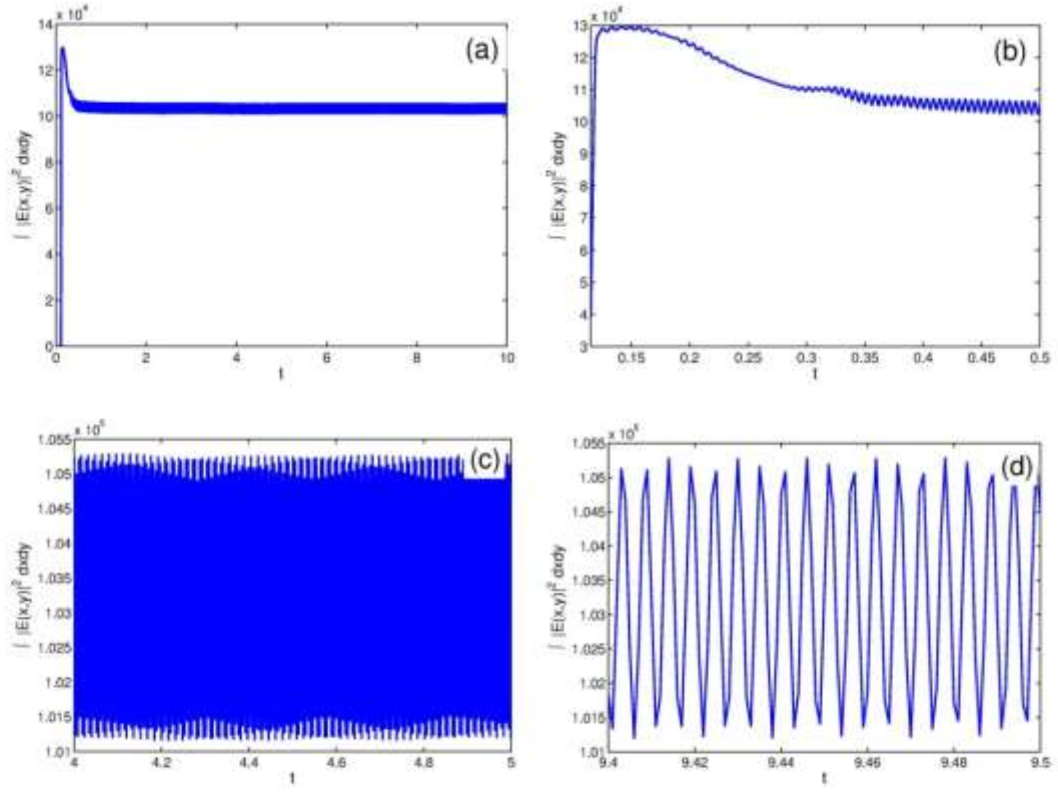

Fig. 2 (Color online.) (a) Evolution of the total energy $\int |E(t,x,y)|^2 dxdy$ for $\beta = 5$. The initial modulation stage ($t < 0.1$) leading to the collapse is too small to be visible here (but see Fig. 3). The panel (b) shows that collapse and viscous heating occur during $0.1 < t < 0.12$, resulting in the rapid increase of the total energy. The panels (b) to (d) are for different sized time segments and thus of different resolutions, demonstrating the modulated oscillations in the total energy. The latter are due to the feedback control and do not appear when $\beta = 0$.

Figure 2(a) shows the evolution of the total energy $\int |E(t,x,y)|^2 dxdy$ for $\beta = 5$. The panels (b) – (d) are for different time segments at different resolutions, showing the modulated oscillations in the total energy arising from the feedback control. The steep energy gain initiated by the collapse beginning at around $t \sim 0.1$ is due to the intense viscous heating, which is inversely proportional to the square of the wavelength. The saturation is due to the $|E|^4$-dependent but scale-independent nonlinear damping. Note that the asymptotic state is already reached before $t = 1$.

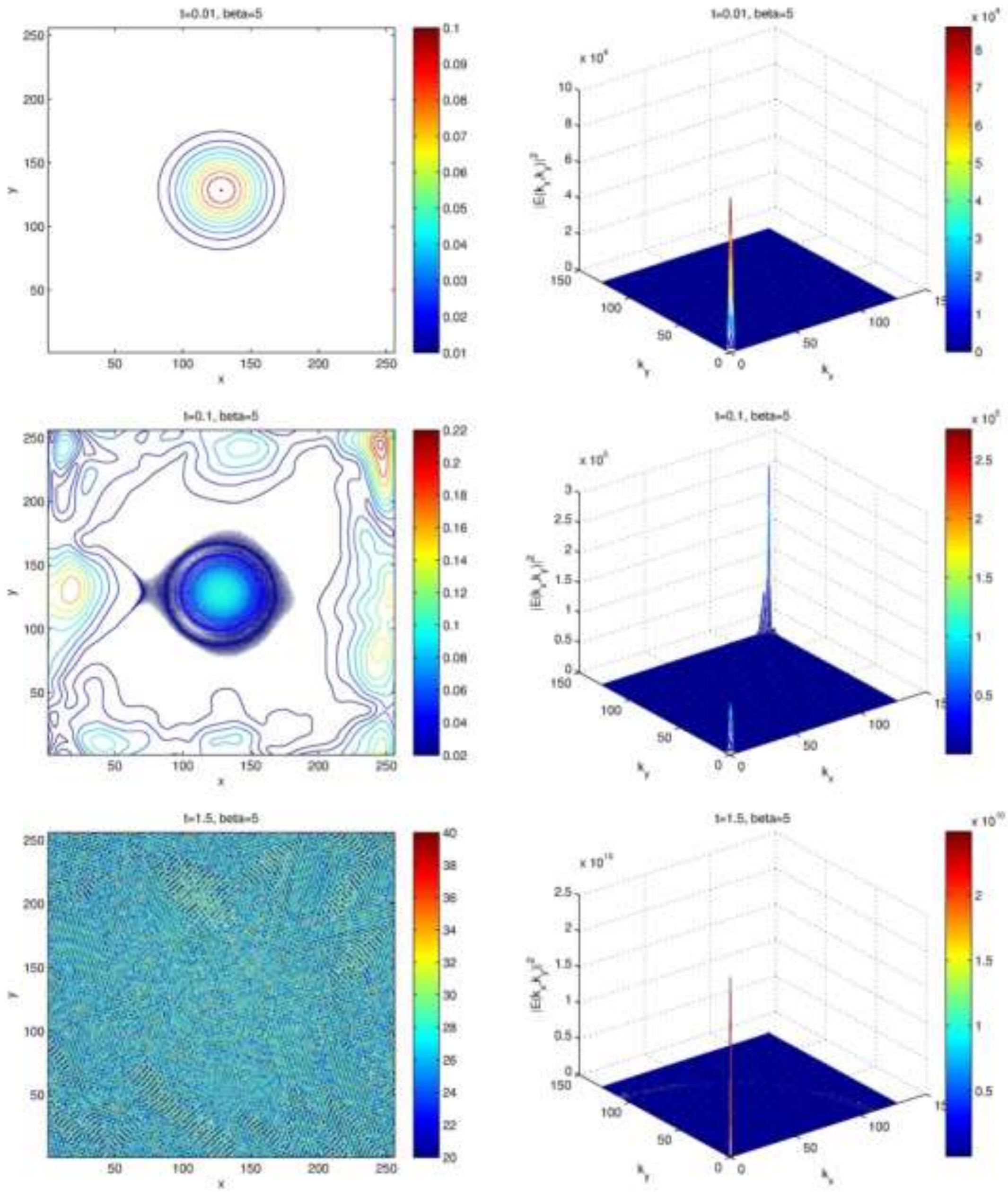

Fig. 3 (Color online). The real space $\left|E(t,x,y)\right|^2$ and a quadrant of the corresponding spectrum $\left|E\left(t,k_x,k_y\right)\right|^2$ (arbitrary units) of the slightly modulated initial pulse at $t=0.01$ (top row), the collapsing stage at $t=0.1$ (center row), and the condensed state at $t=1.5$ (bottom row). Note the large difference in the color bar scales. Except for the last stage, the evolution is similar to that in Ref. 15, where the feedback control is absent. The real-space axes have been redefined (as compared to that in Fig. 1) for convenience of computation.

The left and right columns in Figs. 3 show the full physical space of $\left|E\left(t,x,y\right)\right|^2$ and a quadrant of the corresponding spectrum $\left|E\left(t,k_x,k_y\right)\right|^2$ at different stages of the evolution. The upper row for $t=0.01$ shows that the initial Gaussian pulse given in Fig. 1(a), corresponding to a Gaussian pulse at the small-$\boldsymbol{k}$ corner of the spectrum, remains apparently unchanged. In the center-left panel for $t\sim 0.1$ one can see extremely small-scale perturbations in the initial pulse, corresponding to the appearance of very-short-wavelength modes in the far (largest $\boldsymbol{k}$) corner of the spectrum $\left|E\left(t,k_x,k_y\right)\right|^2$, showing the collapsing stage. Since in the present model the very-short-wavelength modes are subject to strong viscous heating, the collapse is strongly enhanced by the latter, resulting in the rapid increase of the total energy in Fig. 1(b). The energy distribution in the real space also changes abruptly from regular to chaotic, as well as from highly localized to nearly homogeneous. The bottom row shows the condensation that follows, and the system energy is converted back to the longest wavelength (as determined by the system parameters and the numerics, including the initial and boundary conditions) modes. The contribution of viscous heating ($\propto |\boldsymbol{k}|^2$) is now small, and the quasistationary state is realized through a time and space averaged balance between nonlinear damping and feedback control, as can be seen from (4). The condensed state remains strongly turbulent.

The asymptotic state contains large amplitude longest-wavelength modes and they fluctuate strongly in time and space. Figure 4 for the energy spectrum at $t=20$ and 60 also shows that even at very long times there can still occasionally appear, although very weak, the smallest, as well as an intermediate, wavelength modes. Such intermittent behavior can however be expected of turbulent systems. It is also of interest to mention that the very-weak intermediate-wavelength modes have the same wavelength (or $|\boldsymbol{k}|\sim 100$) as the intense condensate that appears when the feedback control is absent [15].

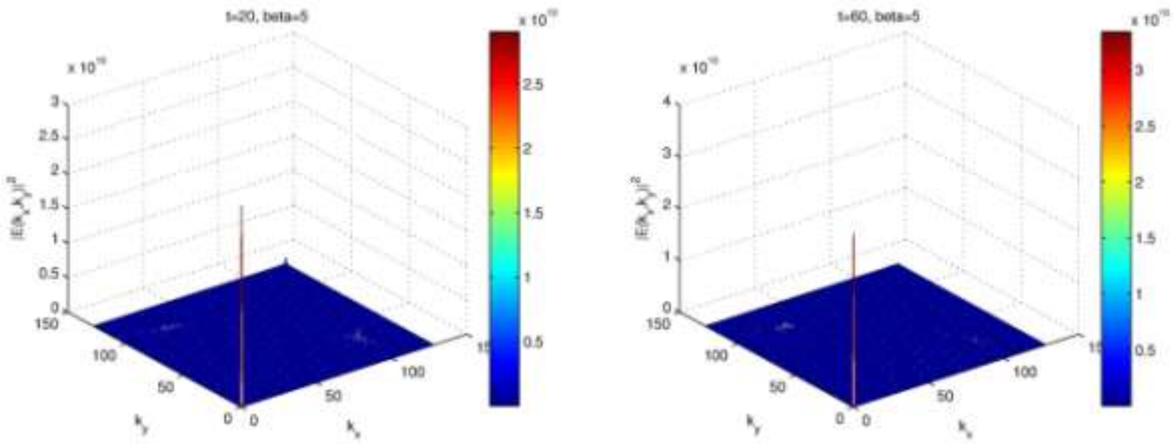

Fig. 4 (Color online.) The energy spectra at $t=20$ and 60. The angle-dependent intermediate ($|\boldsymbol{k}|\sim 100$) and the shortest-wavelength modes can still appear intermittently, but both are

always very weak. (See the online figures if necessary.)

For completeness, in Fig. 5 we present the results for different degrees of feedback control, or β values. One can see that for very small β (= 0.2), modes of a single intermediate wavelength ($|\boldsymbol{k}| \sim 100$) dominate, as found for $\beta = 0$ [15]. This also shows that for the present problem the final state is independent of the fixed external potential, which rapidly becomes negligible (compared to the field energy) when collapse and viscous heating take place. For β = 0.5 and 2.0, the condensation fluctuates rather strongly and can be to modes of two wavelengths, namely the intermediate and the longest. Moreover, even for large β (=20), we still find occasional reappearance (albeit extremely weak) of the shortest wavelength modes created during the collapse. Such intermittent behavior even at long times can be attributed to the stochastic nature of the interacting modes [16].

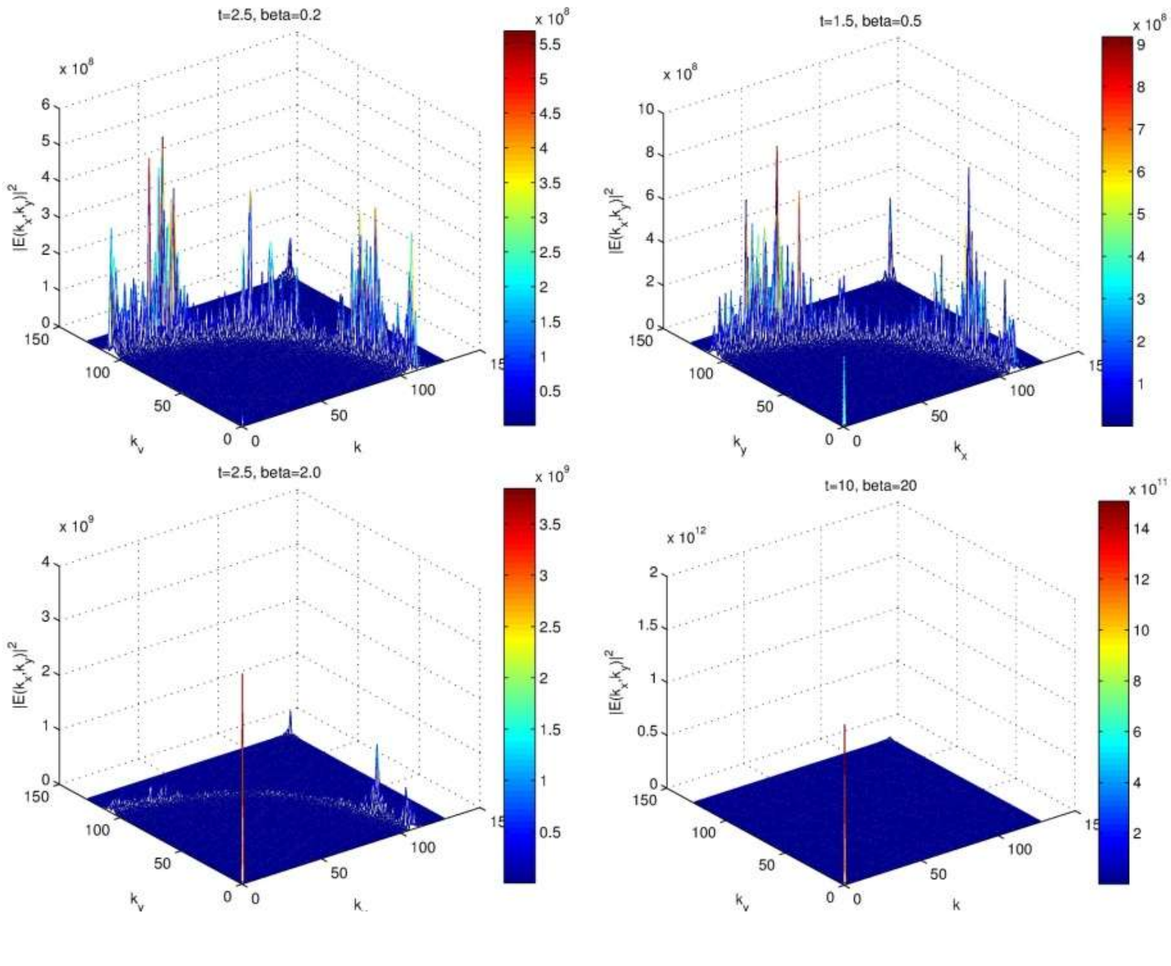

Fig. 5 (Color online.) Effect of feedback control. The asymptotic spectra for $\beta = 0.2$, 0.5,

2.0, and 20, indicating competition among the modes of different wavelengths and angles. The peaks oscillate rapidly in time and space, but the spatial distributions are typical for each case. Note the difference in the color bar scales. (See the online figures if necessary.)

4. Discussion

We have considered a dispersive, diffusive, as well as dissipative nonlinear medium modeled by a modified 2D GNSE including feedback phase and amplitude control. It is found that if the gain and loss of the total system energy can become balanced during the evolution, a localized initial perturbation can suffer modulational instability and collapse into a turbulent state consisting of the smallest-scale modes allowed by the system. Despite the $|\boldsymbol{k}|^2$-dependent viscous heating, when the feedback control is sufficiently strong, the turbulent smallest-scale modes created by the collapse can inverse decay into the largest-scale, also turbulent, modes almost without cascading through the intermediate sized modes. With weaker feedback, one can also find strongly fluctuating asymptotic states with coexisting shortest and longest wavelength modes, as well as an intermediate-wavelength one. In all the cases shown, the post-collapse condensation occurs only in the phase space and the condensed state is always turbulent. No condensation or self-organization in the physical space [16] was found.

Although the post-collapse condensation can be roughly understood as a result of turbulent mode-mode interaction [13], the actual physics of the interaction remains unclear. In fact, existing analytical theories based on the resonant three-wave interaction and wave-kinetic equation models [22-24] invoking small-amplitude perturbations, weak coupling, and/or the random phase approximation do not apply, since except for the initial modulational instability the amplitude of the fluctuations is large, and non-resonant multi-mode interactions can be involved. Accordingly, dedicated theoretical study and modeling of such turbulent interactions are called for.

Acknowledgments

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