Global theory of beta-induced Alfvén eigenmode excited by energetic ions

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Abstract

The two-dimensional global stability and mode structures of high-n beta-induced Alfvén eigenmodes (BAEs) excited by energetic ions in tokamaks are examined both analytically and numerically, employing the WKB-ballooning mode representation along with the generalized fishbone like dispersion relation. Here, $n \gg 1$ is the toroidal mode number. Our results indicate that (i) the lowest radial bound state corresponds to the most unstable eigenmode, and (ii) the anti-Hermitian contributions due to wave-energetic particle resonance give rise to the twisting radial mode structures.

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I. INTRODUCTION AND MOTIVATION

Understanding Alfvén wave instabilities excited by energetic particles (EPs) is of crucial importance to the performance of burning tokamak plasmas. Since such instabilities are typically driven unstable by the finite EP pressure gradient and via wave-EP resonance, the resultant Alfvén wave electromagnetic fluctuations could often lead to rapid EP losses due to the breaking of toroidal symmetry [1–10]. One of such Alfvén wave instabilities, which has received considerable interest, is the beta-induced Alfvén eigenmode (BAE) [11, 12] and is the focus of the present work.

As BAE instability taps the expansion free energy associated with the EP pressure gradient, it typically has a large toroidal mode number, $n \gg 1$, and renders theoretical analysis based on the WKB-ballooning formalism [13–20] feasible. That is, given the WKB representation of the radial envelope, one first solve the one-dimensional (1D) eigenmode equation along the magnetic field. The resultant dispersion relation can then be regarded as the expression for the WKB radial wave number, $k_r(r) \equiv nq'\theta_k(r)$, where $q(r) = rB_T/RB_\vartheta$ is the tokamak safety factor, B_T and B_ϑ are, respectively, the toroidal and poloidal magnetic fields, r and R, respectively, the minor and major radii, and $q' = dq/dr$. Note that k_r and/or θ_k depend on r due to the radial variations of the equilibrium parameters; e.g., the EP pressure. In the next order, the two-dimensional (2D) global eigenvalue problem is then solved by imposing the appropriate boundary conditions and the corresponding quantization conditions on the WKB phase integral.

Up to now, most of the theoretical analyses on BAE stability have been carried out in the lowest order 1D limit; assuming $\theta_k = 0$ [21–23]. While such analyses do provide the necessary insights to the stability of BAE, only the complete 2D analysis could answer the ultimate stability issue and yield the corresponding radially global mode structures. This is one motivation of the present work. Recent experiment and simulation results have, furthermore, shown that EP effects can twist the radial mode structure [24–26]. The other motivation of the present work is, thus, to provide an analytical understanding of this twisting radial mode structures.

The rest of paper is organized as follows. Section II contains the theoretical model as well as the derivation of the corresponding generalized linear fishbone dispersion relation including the finite-θ_k effects due to the radial envelope variations. The analysis and numerical results for the next-order global stability and mode structures are then presented in Sec. III. Final summary and discussion are given in Sec. IV.

II. THEORETICAL MODEL AND THE GENERALIZED LINEAR FISHBONE DISPERSION RELATION

The theoretical approach of the 2D eigenvalue problem considered here closely follows that given in Refs. [16, 17, 27, 28]. We consider here a high-n BAE in a large aspect-ratio $\epsilon = r/R_0 < 1$ axisymmetric low-β $(= 8\pi P/B^2 \sim \epsilon^2)$ tokamak with shifted circular flux surfaces. To be more specific, we adopt here the $(s,\ \alpha)$ model equilibrium [29] with $s = rq'/q$ the magnetic shear and $\alpha \equiv -R_0 q^2 \beta'$ the normalized pressure gradient. Here, P is the plasma pressure and B the equilibrium magnetic field. The plasma is taken to consist of a core (C) component, providing an isotropic Maxwellian background made of electrons and ions, and an energetic (E) component with anisotropic slowing down distribution function. The formal wavelength and frequency orderings for the case of BAEs resonantly excited by EPs are $\omega \approx \omega_{*pi} \approx \omega_{ti} \approx \mathcal{O}(\epsilon^{1/2})\omega_A$, $k_\vartheta \rho_{Li} \approx \mathcal{O}(\epsilon)$, $k_\vartheta \rho_{LE} \sim \mathcal{O}(\epsilon^{1/2})$. Here, $k_\vartheta = nq/r$ is the wave vector in the poloidal direction, ρ_{Li} and ρ_{LE} are, respectively, the thermal and energetic ion Larmor radii, $\omega_{*pi} = (cT_i/e_iB^2)(\mathbf{k}\times\mathbf{B})\cdot\nabla \ln P_i$ is the thermal ion diamagnetic frequency, $\omega_{ti} = \sqrt{2T_i/m_i}/qR_0$ is the thermal ion transit frequency, $\omega_A = v_A/qR_0$ with v_A being the Alfvén speed, e_i and m_i are, respectively, the ion electric charge and mass, P_i is the thermal ion pressure, and $\mathbf{k}$ is the wavevector.

Since $|k_\perp| \gg |k_\parallel|$ and $\beta \ll 1$, we can neglect the compressional Alfvén waves and use the field variables $\delta\phi$ and $\delta\mathbf{A} \simeq \delta A_\parallel \mathbf{b}$ to describe the electromagnetic fluctuations. As $|k_\perp \rho_{Li}|^2 \ll 1$, we also adopt the long wavelength limit and neglect finite thermal ion Larmor radius effects. This allows us to explicitly solve $\delta A_\parallel$ as a function of $\delta\phi$, using the quasineutrality condition, and to reduce the description to a single field variable, $\delta\phi$ [19, 20].

With $n \gg 1$ assumed here, we may adopt the WKB-ballooning mode representation; $\delta\phi$ can then be expressed as [14, 27]

$$\delta\phi(r,\vartheta,\zeta,t) = A(r)e^{-i\omega t}e^{in\zeta}\sum_m e^{-im\vartheta}\frac{1}{2\pi}\int_{-\infty}^{+\infty}\delta\hat{\phi}(\eta)e^{-i\eta(nq-m)}d\eta + \text{c.c.}, \tag{1}$$

where $(r,\ \vartheta,\ \zeta)$ are a straight field line coordinates [19, 20], $-\pi \leq \vartheta \leq \pi$, and η is the coordinate referred to as the extended poloidal angle in the theory of ballooning modes [14].

$\delta\hat{\phi}(\eta)$ is the Fourier transform of $\delta\phi(nq-m)$; i.e., of the radial structure of poloidal harmonics, which is nearly invariant under radial translations by the distance between mode rational surfaces, $\Delta r_n = (nq')^{-1}$, and multiples of it [18]. The radial envelope function $A(r)$ has characteristic spatial dependences on meso-scales, intermediate between the perpendicular wave-length and the equilibrium scale-length. Thus, it is possible to employ the WKB approximation [16, 17], i.e., $A(r) = \sum_k A_k e^{in\int^r \theta_k(r)dq}$, for simplifying the algebra. Here, $\theta_k \equiv k_r/nq'$ with k_r being the WKB wavenumber of the radial envelope; and θ_k is, in general, complex, with $\theta_k = \theta_{kr} + i\theta_{ki}$. Note that the present study differs from that in Ref. [23] where $\theta_k = 0$, which, as will be shown later, corresponds to the most unstable mode.

We assume massless electron response and, for the sake of simplicity, one thermal and one energetic ion species with unit electric charge e that are described by gyrokinetic equation. Following the theory of 2D eigenmodes [16, 21, 30], the lowest-order vorticity equation for the BAE in the presence of energetic ions is given by

$$\left[\frac{\partial^2}{\partial\eta^2}+\Lambda^2-V\right]\delta\hat{\Psi}(\eta)-f^{-1/2}\left\langle\frac{4\pi e_E q^2 R_0^2}{k_\vartheta^2 c^2}J_0(\lambda_{\rho_E})\omega\omega_{dE}\delta K_E\right\rangle = 0, \tag{2}$$

where $V = (s-\alpha\cos\eta)^2/f^2 - \alpha\cos\eta/f$, $f(\eta) \cong 1 + [s(\eta-\theta_k^{(0)}) - \alpha\sin\eta]^2$, $\alpha = \alpha_C + \alpha_E$, $\alpha_C \equiv -R_0 q^2\beta'$, $\alpha_E = -\frac{1}{2}R_0 q^2 \frac{d}{dr}(\beta_{E\parallel} + \beta_{E\perp})$, $\delta\hat{\Psi}(\eta) = f^{1/2}\delta\hat{\phi}(\eta)$, $\omega_{dE} = k_\vartheta \Omega_{dE} g(\eta)$, $\Omega_{dE} = (v_\perp^2/2 + v_\parallel^2)/\omega_{cE}R_0$, $g(\eta) \cong \cos\eta + [s(\eta-\theta_k^{(0)}) - \alpha\sin\eta]\sin\eta$, $\theta_k^{(0)}$ is the lowest order WKB phase, and, from now on, we replace $\theta_k^{(0)}$ by θ_k for simplicity. J_0 is the Bessel function of the first kind and zero index, with argument $\lambda_{\rho_E} = k_\perp \rho_{LE}$, $k_\perp^2 = k_\vartheta^2 + k_r^2$, $\rho_{LE} = v_{\perp E}/\omega_{cE}$, $\omega_{cE} = q_E B/m_E c$, q_E and m_E are the electric charge and mass of the energetic ions, $\langle ... \rangle = \int d^3v(...)$, and the rest of the notations are standard. The inertia response Λ here, including both thermal ion transit resonances as well as diamagnetic effects, has been derived explicitly in Ref. 21. Of that analysis, we report only the main results in Appendix A for reader's convenience. The ballooning equation, Eq. (2), is an eigenvalue equation under the boundary condition that $\delta\hat{\Psi}(\eta)$ vanishes as $\eta \to \pm\infty$. The non-adiabatic EP response δK_E is determined by the gyrokinetic equation [31–33]

$$\left[\omega_{tE}\frac{\partial}{\partial\eta} - i(\omega-\omega_{dE})\right]\delta K_E = i\frac{e_E}{m_E}QF_{0E}\frac{\omega_{dE}}{\omega}J_0(\lambda_{\rho_E})f^{-1/2}\delta\hat{\Psi}(\eta), \tag{3}$$

where $\omega_{tE} = v_{\parallel E}/qR_0$, $QF_{0E} = (\omega\partial_\varepsilon + \hat{\omega}_{*E})F_{0E}$, $\hat{\omega}_{*E}F_{0E} = \omega_{cE}^{-1}(\mathbf{k}\times\mathbf{b})\cdot\nabla F_{0E}$, and F_{0E} is the EP equilibrium distribution function.

Due to $\delta\hat{\Psi}(\eta)$ characterized by two distinct scales [21, 30, 34], adopting asymptotic analyses, we can rewrite the dispersion relation generated by the boundary value problem in Eq. (2) as

$$D(\omega; r, \theta_k) \equiv i\Lambda - \delta W_f - \delta W_k = 0, \tag{4}$$

where δW_f and δW_k are, respectively, the fluid-like and the EP potential energy, and both of them are the function of θ_k. Note that θ_k doesn't enter into the inertia term Λ since it comes from $\eta \to \infty$. Here, we only report the final expressions of δW_k and δW_f including finite-θ_k modifications. Details are given in Appendix B.

For circulating EPs, the non-adiabatic contribution $\delta W_k(\theta_k)$ is given by

$$\begin{aligned}\delta W_k(\theta_k) = \frac{\pi^2 e^2 q^2 R_0^2}{2mc^2} \Bigg\langle \frac{Q F_0 \Omega_d^2}{\omega^2 - \omega_{tE}^2} \Bigg[& \frac{1}{\Delta(1+\Delta^2)^{3/2}|s|} \\ & - \frac{e^{-\frac{2(1+\Delta^2)^{1/2}}{\Delta|s|}} \left[2 + 2\Delta^2 + \Delta(1+\Delta^2)^{1/2}|s|\right]}{\Delta^2(1+\Delta^2)^2 s^2} \left(\cos 2\theta_k + \frac{i\omega_{tE}}{\omega}\sin 2\theta_k\right)\Bigg]\Bigg\rangle.\end{aligned} \tag{5}$$

Here only the dominant transit resonance is considered and the Padé approximation is adopted for Bessel functions [30, 34–36], i.e., $J_0^2(\lambda_{\rho E})J_1^2(\lambda_{dE})/\lambda_{dE}^2 \approx 1/[2(1+\Delta^2 k_\perp/k_\theta^2)]^2$ with $\Delta^2 \equiv k_\vartheta^2(\rho_{LE}^2 + \rho_{dE}^2/2)/4$. Our expression can, thus, recover, in the $\theta_k = 0$ and $|s|$, $\alpha < 1$ limits, the previously obtained result [30, 34]. In the present work, we mainly focus on $\Delta \ll 1$, since $\epsilon < k_\vartheta \rho_{dE} < 1$, which corresponds to the most unstable wavenumber range [30, 34].

The expression of $\delta W_f(\theta_k)$ is given by

$$\delta W_f(\theta_k) \simeq \frac{\pi}{4|s|}\left[s^2 - \frac{3}{2}\alpha^2|s| + \frac{9}{32}\alpha^4 - \frac{5}{2}\alpha e^{-1/|s|}\cos\theta_k - \frac{5\alpha^2}{2|s|}e^{-2/|s|}\cos 2\theta_k\right], \tag{6}$$

where the trial function $\delta\hat{\Psi}(\eta) = 1 + \alpha\cos\eta/f$ is employed in the δW_f calculation to yield higher accuracy for $\alpha \sim |s| \sim \mathcal{O}(1)$.

With the explicit form of $D(\omega; r, \theta_k)$, we can adopt the standard WKB theory [37] to find an approximate solution for $A(r)$.

III. THE GLOBAL EIGENMODE ANALYSIS AND NUMERICAL RESULTS

In this section, we investigate the global BAE stability properties and mode structures. For simplicity, we take F_{0E} to be a single-pitch-angle slowing-down beam ion distribution; i.e., $F_{0E} = N(r)\varepsilon^{-3/2}\delta(\lambda - \lambda_0)$ with the normalization coefficient $N(r) =$

$\sqrt{2(1-\lambda_0 B_0)}B_0\beta_E(r)/2^5\pi^2 m_E\varepsilon_b$. Here, $\beta_E(r)\equiv 2\mu_0 P_E(r)/B_0^2$, $\delta(x)$ is the δ-function, λ_0 is the energetic ion birth pitch angle, $\varepsilon\leq\varepsilon_b$ with ε_b being the EP birth energy per unit mass. Substituting into Eq. (5) and keeping the leading-order contribution, we have

$$\delta W_k=\frac{\pi\alpha_E(r)}{2\sqrt{2}}\bar{\omega}\left[2-\bar{\omega}\ln\left(\frac{\bar{\omega}+1}{\bar{\omega}-1}\right)\right], \tag{7}$$

where $\bar{\omega}=\omega/\omega_{tm}$ and ω_{tm} being the EP transition frequency at the maximum particle energy. Furthermore, $\alpha_E(r)$, the normalized EP pressure gradient, is of the following form:

$$\alpha_E(r)=\alpha_{E0}\exp\left[-\frac{(r-r_0)^2}{L_{PE}^2}\right]\simeq\alpha_{E0}\left[1-\frac{(r-r_0)^2}{L_{PE}^2}\right].$$

Here, $\alpha_{E0}\equiv q^2R_0\beta_E'(r_0)$, L_{PE} is the characteristic α_E scale length and r_0 is the reference mode rational surface; $nq(r_0)=m_0$. Note that, consistent with the assumption that the radial localization is due to EP, the radial dependence of the dispersion function Eq. (4) enters via $\alpha_E(r)$.

Let us assume, $|\theta_{ki}/\theta_{kr}|\ll 1$, to be verified a posteriori. The solution of the dispersion function Eq. (4) yields the condition of the most unstable BAE excited by EPs. The local equilibrium parameters are $r_0=0.5$, $q=2.0$, $s=0.3$, $\epsilon=0.3$, $\beta_i=0.01$, $\omega_{*ni}/\omega_{ti}=0.1$, $\eta_i=2.0$, $L_{Pi}=0.9$, $\eta_E=0.0$, $\lambda_0 B_0=0.0$, $L_{PE}=0.3$, $\upsilon_{Ei}=4.0$, $n_{Ei}=0.03$, and $n=5$. Here, L_{Pi} is the pressure gradient scale length of thermal ions. The contour plot of the growth rate $\mathrm{Im}(\omega/\omega_{ti})$ in $(r,\,\theta_k)$ plane is presented in Fig. 1. It is obvious that the most unstable mode occurs around $r=r_0$ and $\theta_k=0$.

In this case, we may Taylor expand the dispersion function $D(\omega;r,\theta_k)$ for $(r,\,\theta_k)$ around $(r_0,\,0)$, respectively. Keeping the leading-order contributions, Eq. (4) then becomes

$$\theta_k^2=Q(\omega;r),\quad Q(\omega;r)=2(i\Lambda-\delta W_{f0}-\delta W_k)/\delta W_f^{(2)}; \tag{8}$$

where

$$\delta W_{f0}=\delta W_f(\theta_k=0)=\frac{\pi}{4|s|}\left[s^2-\frac{3}{2}\alpha^2 s+\frac{9}{32}\alpha^4-\frac{5}{2}\alpha e^{-1/|s|}\right],$$

and

$$\delta W_f^{(2)}=\left.\frac{\partial^2\delta W_f}{\partial\theta_k^2}\right|_{\theta_k=0}=\frac{5\pi}{8|s|}\alpha e^{-1/|s|}.$$

Equation (8) defines two branches, θ_{k+} and θ_{k-}, which lead to the WKB solutions

$$A_+e^{i\int^r nq'\theta_{k+}dr'}\qquad\text{and}\qquad A_-e^{i\int^r nq'\theta_{k-}dr'},$$

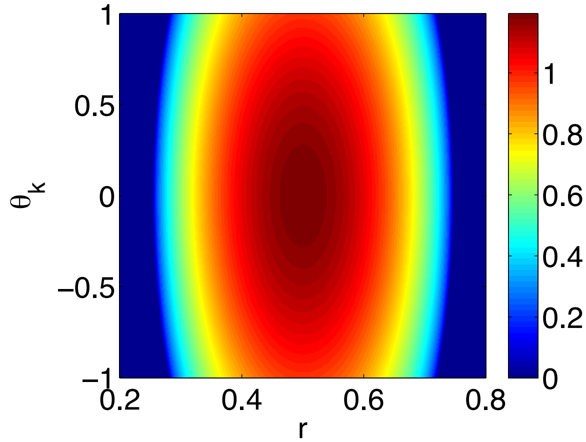

FIG. 1. The contour plot of the growth rate $\mathrm{Im}(\omega/\omega_{ti})$ of $D(\omega; r, \theta_k) = 0$. Fixed parameters are $r_0 = 0.5$, $q = 2.0$, $s = 0.3$, $\epsilon = 0.3$, $\beta_i = 0.01$, $\omega_{*ni}/\omega_{ti} = 0.1$, $\eta_i = 2.0$, $L_{Pi} = 0.9$, $\eta_E = 0.0$, $\lambda_0 B_0 = 0.0$, $L_{PE} = 0.3$, $\upsilon_{Ei} = 4.0$, $n_{Ei} = 0.03$, and $n = 5$.

valid except near the turning points. There, A_+ and A_- are related via the WKB connection formula [38]. Defining $z \equiv |nq'|(r - r_0)$ and, thus $\int^r nq'\theta_k dr' = \int^z \theta_k dz'$. The global dispersion relation for the BAE excited by EPs is then given by the following quantization condition;

$$\int_{z_{T-}}^{z_{T+}} \sqrt{Q(\omega; z)} dz = (L + 1/2)\pi \quad \text{L=0, 1, 2, ... ,} \tag{9}$$

where $z_{T\pm}$ are the (complex) two regular WKB turning points, defined by the condition $Q(\omega; z) = 0$; while the integer L is the radial mode number denoting the different eigenmodes.

Numerical solutions of Eq. (9) have been obtained for the same parameters of Fig. 1. Results are presented in Fig. 2, which shows that, unlike the frequencies, below the accumulation point $\omega/\omega_{ti} = 5.2326$ [23], the corresponding growth rates are strongly dependent on the eigenmode number L. The lower L modes are more unstable due to the sharp localization of the EPs. In addition, for the most unstable mode, $|\theta_{ki}/\theta_{kr}| \sim |\mathrm{Im}(\omega/\omega_{\mathrm{ti}})/\mathrm{Re}(\omega/\omega_{\mathrm{ti}})| \sim \mathcal{O}(10^{-1})$; self-consistently verifying Taylor expansion around $\theta_k = 0$ a posteriori.

Based on above analysis, the complete perturbation $\delta\phi(r, \vartheta, \zeta, t)$ can then be rewritten as

$$\begin{aligned} \delta\phi(r, \vartheta, \zeta, t) = e^{-i\omega t} e^{in\zeta} \sum_m \Big\{ & A_+ e^{i\int^r nq'\theta_{k+} dr'} e^{-im\vartheta} \frac{1}{2\pi} \int_{-\infty}^{\infty} d\eta e^{-i(nq-m)\eta} \delta\hat{\phi}_+(\eta) \\ & + A_- e^{-i\int^r nq'\theta_{k-} dr'} e^{-im\vartheta} \frac{1}{2\pi} \int_{-\infty}^{\infty} d\eta e^{-i(nq-m)\eta} \delta\hat{\phi}_-(\eta) \Big\} + \text{c.c.}, \end{aligned} \tag{10}$$

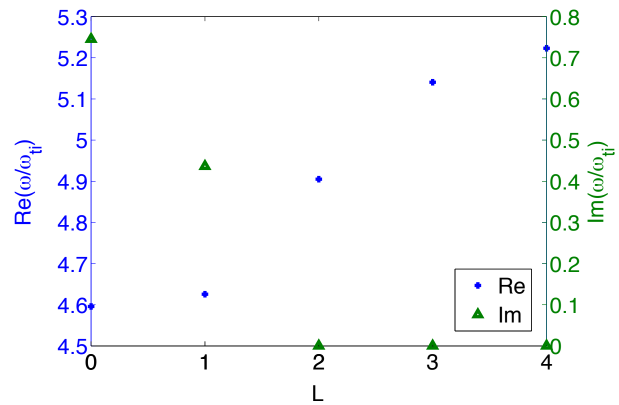

FIG. 2. Frequencies (circles) and growth rate (triangles) vs L. Here, the parameters are the same as those of Fig. 1.

where $\delta\hat{\phi}_+(\eta)$ and $\delta\hat{\phi}_-(\eta)$ are, respectively, the solutions of Eq. (2) corresponding to θ_{k+} and θ_{k-}, where $\theta_{k-} = -\theta_{k+}$. Suppressing the time dependence and taking $\zeta = 0$, we can further reduce Eq. (10) to

$$\delta\phi(r,\vartheta) = \sum_m e^{-im\vartheta}\left[A_+ e^{i\tilde{\Theta}(z)}\delta\phi_+(z) + A_- e^{-i\tilde{\Theta}(z)}\delta\phi_-(z)\right] + \text{c.c.}, \tag{11}$$

where $z \equiv nq(r) - m$, $A_+ = -iA_-$ for $|z| \leq |z_T|$ where the mode is predominantly localized, $\delta\phi_\pm(z) = 1/2\pi\int_{-\infty}^{\infty} d\eta\delta\hat{\phi}_\pm(\eta)e^{-iz\eta}$, and $\tilde{\Theta}(z) \equiv \int_{z_{T-}}^{z}\theta_{k+}(z')dz'$. Note, for $n \gg 1$, we have formally $|\theta_k| \sim \mathcal{O}(n^{-1/2})$, $|z_T| \sim \mathcal{O}(n^{1/2})$, $|\tilde{\Theta}(z)| \sim \mathcal{O}(1)$, and $|\delta\phi_\pm(z)|$ has mode width of $|z| \sim O(1)$.

According to Eq. (9), by symmetry considerations, we have $\int_{z_{T-}}^{0}\theta_{k+}(z)dz = \pi/4$ for the most unstable case ($L = 0$). Thus, Eq. (11) can further be reduced to

$$\delta\phi(r,\vartheta) = \frac{C}{\theta_{k+}^{1/2}(z)}\sum_m e^{-im\vartheta}\left[e^{i\Theta(z)}\delta\phi_+(z) + e^{-i\Theta(z)}\delta\phi_-(z)\right] + \text{c.c.}, \tag{12}$$

where C is a constant and $\Theta(z) \equiv \int_0^z \theta_{k+}(z')dz'$.

With these results, we can theoretically analyze the global mode structure. In Eq. (12), neglecting effects due to finite $|\theta_k| \sim \mathcal{O}(n^{-1/2}) \ll 1$ in $\delta\phi_\pm(z)$, we have $\delta\phi_\pm(z) \simeq \delta\phi_0(z)$, which is independent of θ_k. If the system is purely Hermitian, e.g., in the ideal MHD limit, both $\delta\phi_0(z)$ and $\Theta(z)$ are real, i.e., $\delta\phi_0(z) = \delta\phi_{0r}(z)$ and $\Theta(z) = \Theta_r(z)$. We can then write Eq. (12) as

$$\delta\phi(r,\vartheta) \propto \frac{1}{\theta_{k+}^{1/2}(z)}\sum_m \{\cos[m\vartheta - \Theta_r(z)] + \cos[m\vartheta + \Theta_r(z)]\}\,\delta\phi_{0r}(z). \tag{13}$$

Since $\delta\phi(r,\vartheta) = \delta\phi(r,-\vartheta)$, we see that the poloidal mode structure is up-down symmetric. For fixed r, the mode amplitude peaks at $\frac{\partial\delta\phi}{\partial\vartheta}\big|_{\vartheta=\vartheta_*} = 0$, i.e., $\sin(m\vartheta_*)\cos(\Theta_r(z)) = 0$. Thus, the mode amplitude of the Hermitian system peaks at $m\vartheta_* = j\pi$ for all z (or r), j being an integer. In the following analysis, without loss of generality, we just focus on $j = 0$.

In the general case with finite anti-Hermitian part of $D(\omega; r, \theta_k)$, e.g., when the EP wave-particle resonances are included, we can write $\delta\phi_0(z) = \delta\phi_{0r}(z) + i\delta\phi_{0i}(z)$ with $|\delta\phi_{0i}(z)/\delta\phi_0(z)| \sim \mathcal{O}(\alpha_E) < 1$ and $\Theta(z) = \Theta_r(z) + i\Theta_i(z)$ with $\Theta_i(z) \sim \mathcal{O}(\alpha_E)$. Assuming $1 > |\alpha_E| > \theta_k \sim \mathcal{O}(n^{-1/2})$, i.e., concentrating on the wave-particle resonant effect, we can write Eq. (12) as

$$\begin{aligned}\delta\phi(r,\vartheta) \propto \frac{e^{-\Theta_i(z)}}{\theta_{k+}^{1/2}(z)} \sum_m \Big\{ & \big[\cos(m\vartheta - \Theta_r(z)) + e^{2\Theta_i(z)} \cos\big(m\vartheta + \Theta_r(z)\big)\big]\delta\phi_{0r}(z) \\ & + \big[\sin\big(m\vartheta - \Theta_r(z)\big) + e^{2\Theta_i(z)} \sin\big(m\vartheta + \Theta_r(z)\big)\big]\delta\phi_{0i}(z)\Big\}.\end{aligned} \tag{14}$$

Note that $|\Theta_i(z)| \sim \mathcal{O}(\alpha_E) < 1$, Eq. (14) can be further reduced to

$$\begin{aligned}\delta\phi(r,\vartheta) \propto & \frac{e^{-\Theta_i(z)}}{\theta_{k+}^{1/2}(z)} \sum_m \Big\{ \big[\cos\big(m\vartheta - \Theta_r(z)\big) + \cos\big(m\vartheta + \Theta_r(z)\big)\big]\delta\phi_{0r}(z) \\ & + 2\Theta_i(z)\cos\big(m\vartheta + \Theta_r(z)\big)\delta\phi_{0r}(z) \\ & + \big[\sin\big(m\vartheta - \Theta_r(z)\big) + \sin\big(m\vartheta + \Theta_r(z)\big)\big]\delta\phi_{0i}(z)\Big\}.\end{aligned}$$

The value of ϑ_*, where the mode amplitude peaks, is then given by

$$\begin{aligned}&\big[\sin\big(m\vartheta_* - \Theta_r(z)\big) + \sin\big(m\vartheta_* + \Theta_r(z)\big)\big]\delta\phi_{0r}(z) + 2\Theta_i(z)\sin\big(m\vartheta_* + \Theta_r(z)\big)\delta\phi_{0r}(z) \\ &\qquad - \big[\cos\big(m\vartheta_* - \Theta_r(z)\big) + \cos\big(m\vartheta_* + \Theta_r(z)\big)\big]\delta\phi_{0i}(z) = 0.\end{aligned} \tag{15}$$

Expanding $\vartheta_* \simeq \vartheta_{*0} + \vartheta_{*1}$, we then recover the Hermitian result in the lowest order. In the next order,

$$m\vartheta_{*1}\cos(\vartheta_{*0})\cos(\Theta_r(z))\delta\phi_{0r}(z) + \Theta_i(z)\sin\big(\Theta_r(z)\big)\delta\phi_{0r}(z) - \cos\big(\Theta_r(z)\big)\delta\phi_{0i}(z) = 0. \tag{16}$$

It follows that

$$m\vartheta_{*1}(z) \simeq -\Theta_i(z)\tan\big(\Theta_r(z)\big) + \frac{\delta\phi_{0i}(z)}{\delta\phi_{0r}(z)}. \tag{17}$$

Noting $\delta\phi_0(-z) = \delta\phi_0(z)$ and $\Theta(-z) = \Theta(z)$, we then have $\vartheta_{*1}(-z) = \vartheta_{*1}(z)$ and $\vartheta_{*1}(z) \neq 0$ generally. That is, finite $\Theta_i(z)$ and $\delta\phi_{0i}(z)$, resulting from the wave-particle resonance contribution to the anti-Hermitian part of $D(\omega; r, \theta_k)$, can twist the radial mode structure away from $m\vartheta_* = 0$; resulting in the loss of up-down symmetry in the global mode structure.

In order to numerically illustrate the anti-Hermitian component effects, we investigate numerically the BAE poloidal mode structure in two limits; i.e., (i) the ideal MHD limit in the absence of EP; (ii) the BAE excited by non-perturbative EP via wave-particle resonance interaction. In the first case, we assume that the radial dependence of the dispersion function $D(\omega, r, \theta_k)$ comes from the fluid-like term $\delta W_f(r, \theta_k)$ with $\delta W_f(r, \theta_k) \simeq \delta W_f(\theta_k)\big[1 - \frac{(r-r_0)^2}{L_{Pi}^2}\big]$. The inertia term Λ_I adopted here corresponds to the MHD limit ($\omega \gg \omega_{ti}$) of Eq. (A1) along with $\omega \gg \omega_{*pi}$ and neglecting damping. We then have $\Lambda_I^2 = \frac{\omega^2}{\omega_A^2}\left[1 - q^2\frac{\omega_{ti}^2}{\omega^2}\left(\frac{7}{4}+\tau\right)\right]$ [12, 21]. In this case, θ_k^2 is given by

$$\theta_k^2 = \frac{2}{\delta W_f^{(2)}}\left\{i\Lambda_I - \delta W_{f0}\left[1 - \frac{(r-r_0)^2}{L_{Pi}^2}\right]\right\}. \tag{18}$$

Adopting the trial function $\delta\hat{\Psi}_\pm(\eta) \simeq e^{i\Lambda\eta}$ with $\Lambda = \Lambda_r + i\Lambda_i$ and applying the inverse Fourier transform, Eq. (12) can be expressed as

$$\delta\phi(r,\vartheta) = \frac{C}{\theta_{k+}^{1/2}(z)s\pi}\sum_m e^{-im\vartheta}\left[e^{i\Theta(z)} + e^{-i\Theta(z)}\right]K_0\big(\sqrt{-(\Lambda_i - i\Lambda_r + iz)^2}/s\big) + \text{c.c.}, \tag{19}$$

where $K_0(x)$ is the modified Bessel function of the second kind and zero index.

The poloidal mode structures of $\delta\phi(r,\vartheta)$ given by Eq. (19) are plotted in Fig. 3. The upper panel corresponds to the marginally unstable ideal MHD BAE case and the lower panel to the BAE excited by EP case. The left column, Figs. 3(a) and (c) are plots in the (R, Z) plane; while the right column shows the magnified mode structures in (r, ϑ) plane. The blue dashed lines shown in Fig. 3(b) and (d), meanwhile, represent the locations of the mode amplitude peaks theoretically predicted by Eq. (17). Here, $m = 9$, 10, and 11, $L = 0$, and other parameters are the same as those of Fig. 1.

Figure 3 shows that the up-down symmetry of the ideal MHD mode structures (Fig. 3(a) and (b)) is broken by the anti-Hermitian component of the dispersion function due to the wave-EP resonant interaction, and exhibits the twisting 2D mode structures (Fig. 3(c) and (d)). The analytically predicted peak agrees well with that of the contour plot. We note that the present theoretical results are also consistent with the numerical simulations [24, 25] and experiment [39].

IV. SUMMARY AND DISCUSSION

In this paper we have investigated the global stability properties and mode structures of the high-n BAEs excited by a radially localized source of energetic ions in tokamaks.

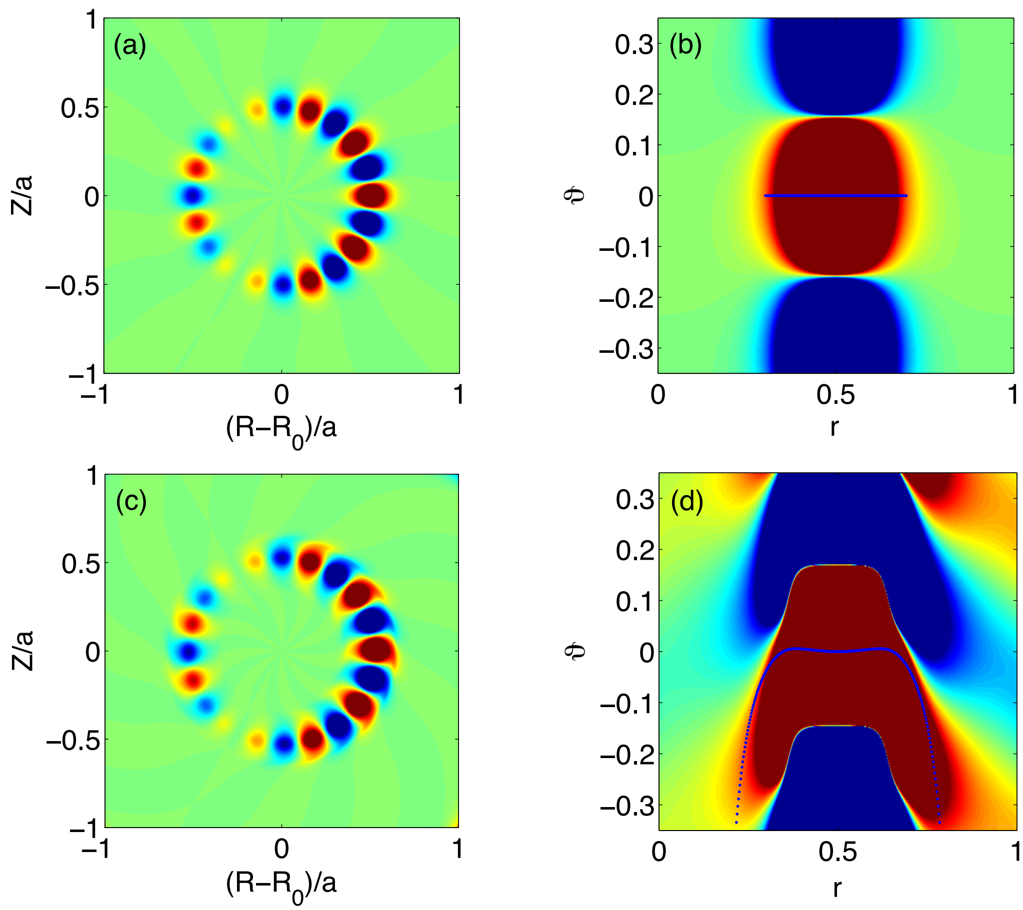

FIG. 3. Poloidal contour plots of electrostatic potential $\delta\phi(r,\vartheta)$ corresponding to the ideal-MHD limit ((a) and (b)) as well as the BAE excited by EP ((c) and (d)), respectively. Here, $m = 9$, 10, and 11, $L = 0$, and other parameters are the same as those of Fig. 1.

Our analysis employs the $1/n$ expansion and the radially WKB approximation. The lowest order expansion corresponds to local stability and mode structure along the field line and is solved via asymptotic variational analysis. The next order yields global stability and mode structures via the quantization condition.

Analytical and numerical analyses demonstrate that, when the non-perturbative EP wave-particle resonant effect is considered, (i) the BAE is radially localized by the EP-pressure-gradient drive and the lowest bound state is most unstable, and (ii) the BAE exhibits the typical twisting radial mode structure in contrast to the ideal MHD limit with up-down symmetry. The present results offer specific theoretical explanations for the experimental and numerical simulation observations of the asymmetric mode structures in the poloidal plane.

The present work, for the sake of simplicity, focuses on the anti-Hermitian effects due to wave-particle resonant interactions. Other effects such as equilibria with sheared flow, Landau damping as well as radial electric field are left out and need to be included in future studies.

In addition, we note that this work is based on the approximation of $|\theta_k| \ll 1$, which, indeed, greatly simplifies our analysis. However, in some parameter regimes of interest, large $|\theta_k|$ will be closely related to parallel momentum transport [40–47]. It is, thus, also desirable to extend the present analysis to the regime of $|\theta_k| \sim 1$.

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Appendix A: The explicit expression of the generalized inertia term, Λ, appearing in Eq. (2)

This part closely follows Ref. 21.

$$\Lambda^2 = \frac{\omega^2}{\omega_A^2}\big(1-\frac{\omega_{*pi}}{\omega}\big) + q^2\frac{\omega\omega_{ti}}{\omega_A^2}\Big[\big(1-\frac{\omega_{*ni}}{\omega}\big)F(\omega/\omega_{ti}) - \frac{\omega_{*Ti}}{\omega}G(\omega/\omega_{ti}) - \frac{N^2(\omega/\omega_{ti})}{D(\omega/\omega_{ti})}\Big], \tag{A1}$$

with ω_{*ni} and ω_{*Ti} being, respectively, the thermal ion diamagnetic frequencies due to density and temperature gradient only. Furthermore, the functions $F(x)$, $G(x)$, $N(x)$ and $D(x)$, with $x = \omega/\omega_{ti}$, $\tau = T_e/T_i$, and using the plasma dispersion function $Z(x)$, are defined as

$$\begin{aligned}
Z(x) &= \pi^{-1/2}\int_{-\infty}^{\infty}\frac{e^{-y^2}}{y-x}dy \\
F(x) &= x(x^2+3/2) + (x^4+x^2+1/2)Z(x) \\
G(x) &= x(x^4+x^2+2) + (x^6+x^4/2+x^2+3/4)Z(x) \\
N(x) &= \big(1-\frac{\omega_{*ni}}{\omega}\big)[x+(1/2+x^2)Z(x)] - \frac{\omega_{*Ti}}{\omega}[x(1/2+x^2)+(1/4+x^4)Z(x)] \\
D(x) &= \big(\frac{1}{x}\big)\big(1+\frac{1}{\tau}\big) + \big(1-\frac{\omega_{*ni}}{\omega}\big)Z(x) - \frac{\omega_{*Ti}}{\omega}[x+(x^2-1/2)Z(x)].
\end{aligned} \tag{A2}$$

Appendix B: Detailed derivation of $\delta W_k(\theta_k)$

δW_k represents EPs nonadiabatic contribution; and its the simplest expression for circulating particles is given by [34]

$$\delta W_k = \frac{2\pi e q^2 R_0^2}{c^2}\omega \int_{-\infty}^{\infty} d\eta \left\langle \delta S_d^* \delta K_d \right\rangle_E , \tag{B1}$$

where $\delta S_{dE} = \Omega_d f^{-1/2} g J_0(\lambda_{\rho E})$ and δK_{dE} satisfies Eq. (3) in the drift-center representation, $\delta K_{dE} = \delta K_E e^{i\eta_d}$, $\eta_d = -\lambda_{dE}\cos\eta$, and $\lambda_{dE} = k_\perp \rho_{dE} = k_\perp \Omega_{dE}/\omega_{tE}$. For convenience, we use the shifted variable $\eta' = \eta - \theta_k$. In the $\eta'-$Fourier conjugate space, it can be shown that geodesic curvature is important for $|s\eta'| > 1$ ($\epsilon < k_\vartheta \rho_{dE} < 1$) [17, 28, 30, 34]. Therefore, we have $f^{-1/2} g \simeq \sin(\eta' + \theta_k)$.

Omitting the prime superscript in η' and using the identity $e^{-i\lambda\cos\eta} = \sum\limits_{p=-\infty}^{\infty} i^p (-1)^p J_p(\lambda) e^{ip\eta}$, we have

$$\sin(\eta+\theta_k)\cdot e^{i\eta_d} = \sum_{p=1}^{\infty} 2(-1)^p i^{p+1} \frac{pJ_p(\lambda_{dE})}{\lambda_{dE}} \sin p(\eta+\theta_k). \tag{B2}$$

For $|s|$, $|\alpha| < 1$ it is then reasonable to adopt the following trial functions

$$\delta\hat{\Psi}(\eta) \simeq \begin{cases} 1 & \eta \lesssim 1/s \\ e^{i\Lambda\eta} & \eta \gtrsim 1/\Lambda \end{cases} .$$

Then, Eq. (3) can be reduced to

$$(\omega_{tE}\partial_\eta - i\omega)\delta K_{dE} = i\tfrac{e}{m}\tfrac{QF_0}{\omega} \sum_{p=1}^{\infty} \delta S_{dp} \sin p(\eta+\theta_k), \tag{B3}$$

where

$$\delta S_{dp} = \Omega_{dE} J_0(\lambda_{\rho E}) 2(-1)^p i^{p+1} \frac{pJ_p(\lambda_{dE})}{\lambda_{dE}}.$$

From Eq. (B3), we obtain

$$\delta K_{dE} = \frac{e}{m}\frac{QF_0}{\omega} \sum_{p=1}^{\infty} \frac{\delta S_{dp}}{p^2\omega_{tE}^2 - \omega^2}\left[\sin p(\eta+\theta_k) - \frac{ip\omega_{tE}}{\omega}\cos p(\eta+\theta_k)\right]. \tag{B4}$$

Finally, substituting Eq. (B4) into Eq. (B1), we obtain

$$\begin{aligned} \delta W_k(\theta_k) &= \frac{2\pi e^2 q^2 R_0^2 \omega}{mc^2} \left\langle \sum_{p=1}^{\infty} \frac{QF_0}{p^2\omega_{tE}^2 - \omega^2} \int_{-\infty}^{\infty} d\eta \left|\delta S_{dp}\right|^2 \right\rangle \\ &= \frac{8\pi e^2 q^2 R_0^2}{mc^2} \sum_{p=1}^{\infty} \left\langle \frac{QF_0}{p^2\omega_{tE}^2 - \omega^2} \int_{-\infty}^{\infty} d\eta \frac{J_0^2(\lambda_{\rho E}) p^2 J_p^2(\lambda_{dE})}{\lambda_{dE}^2} \Big[\sin^2 p(\eta+\theta_k) \right. \\ &\left. - \frac{ip\omega_{tE}}{\omega} \sin p(\eta+\theta_k)\cos p(\eta+\theta_k)\Big] \right\rangle. \end{aligned} \tag{B5}$$

Keeping only the $p = 1$ leading order term and adopting the Padé approximation for Bessel functions, Eq. (B5) can be reduced to Eq. (5).

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