On fast radial propagation of parametrically excited geodesic acoustic mode

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The spatial and temporal evolution of geodesic acoustic mode (GAM) parametrically excited by drift waves (DW) is investigated both analytically and numerically. Our results show that the nonlinearly excited GAM propagates at a group velocity which is, typically, much larger than that due to finite ion Larmor radius as predicted by the linear theory. The nonlinear theory presented here could, thus, explain the discrepancies between the experimentally measured dispersion relation of GAM and that from the linear theory. Further implications of these findings for proper understanding of experimental observations are also discussed.

Geodesic Acoustic Modes (GAM) [1, 2] are finite-frequency components of zonal structures [3, 4] unique in toroidal plasmas, which are capable of scattering microscopic drift-wave-type (DW) turbulence [5] including drift Alfvén waves into stable short radial-wavelength regime [2, 6–9], and, therefore, regulating the turbulence intensity and the associated wave-induced transports [10].

It is known that GAM has a finite linear group velocity due to finite Larmor radius (FLR) effects, and this linear group velocity is typically radially outward in consistence with GAM continuum, due to radial temperature profiles. This linear group velocity of GAM has been discussed in several works [2, 11, 12], and is shown to have important consequences on the nonlinear excitation of GAM by DW turbulences and change the absolute/convective nature of the parametric instability [8, 13]. Radial propagation of GAM has been observed in several experiments [14, 15], and the propagation has been interpreted using the linear theory of kinetic GAM (KGAM) [16]. However, in-depth analysis of the experimentally obtained dispersion relation leads to the fact that, even though a quadratic dependence of GAM frequency on its radial wavevector is indeed obtained, qualitatively consistent with linear theory of KGAM [16], the coefficient for ion FLR effects is much larger than that predicted by linear theory [11, 17]. This discrepancy has also been found in a numerical simulation [12], and up to now, there is no first-principle-theory-based interpretation.

We note that, GAM is an $n = 0/m \simeq 0$ mode such that it cannot be driven unstable by expansion free energy of the plasma. Here, m and n are, respectively, the poloidal and toroidal mode numbers of torus. Thus, GAM can be observed only when it is nonlinearly driven by ambient turbulences, and in this case, the spatial-temporal evolution of GAM is dominated by the nonlinear drive of DW. As a result, the propagation of GAM, and experimental dispersion relation should also be interpreted using nonlinear theory.

The equations describing the nonlinear interactions between GAM and DW turbulence are derived using gyrokinetic theory [2, 8]. Assuming that the DW turbulence constitutes a constant-amplitude pump wave and a lower sideband with a much smaller amplitude due to the modulation of GAM, the normalized coupled nonlinear equations describing GAM excitation by DW are then given as equations (9) and (10) of Ref. 8. Since these two equations are the starting point of this work, we give it explicitly here. Assuming the following mode structure decomposition of the single $n \neq 0$ DW and GAM in the straight-field-line toroidal flux coordinate (r, θ, φ)

$$\begin{aligned} \delta\phi_P &= A_P e^{-in\varphi - i\omega_P t} \sum_m e^{im\theta} \Phi_0(nq - m) + c.c., \\ \delta\phi_S &= A_S e^{in\varphi - i(\omega_G - \omega_P)t} \sum_m e^{-im\theta} \Phi_0^*(nq - m) + c.c., \\ \delta\phi_G &= A_G e^{-i\omega_G t} + c.c.; \end{aligned}$$

and the eikonal Ansatz is assumed for the radial envelopes; i.e.,

$$\begin{aligned} A_P &= e^{i\int k_P dr}, \\ A_S &= e^{-i\int k_P dr} \left(e^{i\int k_G dr} + c.c. \right), \\ A_G &= e^{i\int k_G dr} + c.c.; \end{aligned}$$

the nonlinear equations describing GAM excitation by DW are then [8]

$$\left(\partial_t + \gamma_S + i\omega_P - i\omega_* - iC_d \omega_* \rho_i^2 \partial_r^2 \right) A_S = \Gamma_0^* \mathscr{E}, \tag{1}$$

$$\left(\partial_t (\partial_t + 2\gamma_G) + \omega_G^2 - C_G \omega_G^2 \rho_i^2 \partial_r^2 \right) \mathscr{E} = i\omega_G \Gamma_0 \partial_r^2 A_S. \tag{2}$$

Here, subscripts P, S and G represent, respectively, pump DW wave, lower sideband and GAM, $\Phi_0(nq - m)$ is the fine radial scale structure associated with finite $k_\parallel$ and magnetic shear, $\mathscr{E} \equiv \partial_r A_G/\alpha$ is the electric field of GAM, $\Gamma_0 \equiv (\alpha_i T_i/\omega_P T_e)^{1/2} c k_{\theta,P} A_P/B$ is the normalized pump wave amplitude, $\alpha \equiv i(\alpha_i \omega_P T_e/T_i)^{1/2}$ with $\alpha_i \equiv 1 + \delta P_\perp/(e n_0 \delta\phi_P)$ is an order unity coefficient [18], $\delta P_\perp$ is the perturbed perpendicular pressure due to $\delta\phi_P$ in the $k_\perp \rho_i \ll 1$ limit [18], γ_S and γ_G are the Landau damping rates of DW sideband and GAM, ω_* is the diamagnetic frequency, $\rho_i = mcv_\perp/eB$ is the Larmor radius of ions. The kinetic term in equation (1), i.e., the term proportional to C_d comes from finite radial envelope variation due to the coupling between neighboring poloidal harmonics. The expression for C_d can be derived from equation (19) of Ref. 19, and one has $C_d \sim O(\epsilon/(n^2q'^2\rho_i^2))$ with q being the safety factor. On the other hand, the kinetic term in equation (2); i.e., the term proportional to C_G comes from FLR of GAM. Thus, $C_G \sim O(1)$ and its detailed expression be obtained from equation (9) of Ref. [17]. Other notations are standard. We note that, system nonuniformities in equations (1) and (2), which may affect qualitatively the convective/absolute nature of the parametric process as shown in Ref. 8, are systemically ignored here, since we focus on the radial propagation of the parametrically excited GAM in this work.

Equations (1) and (2) can be solved using two-spatial two-temporal scales expansion of $\mathscr{E}$ and A_S, such that $\partial_t = -i\omega_0 + \partial_\tau$ and $\partial_r = ik_0 + \partial_\xi$, with τ and ξ denoting the slow temporal and spatial variations. The coupled nonlinear equations reduce to

$$(\partial_\tau + V_S\partial_\xi)\,A_S \;=\; \Gamma_0^*\mathscr{E}, \tag{3}$$

$$(\partial_\tau + V_G\partial_\xi)\,\mathscr{E} \;=\; \frac{1}{2}\Gamma_0(k_0^2 - 2ik_0\partial_\xi)A_S. \tag{4}$$

Here, in equation (3) and (4), γ_S and γ_G are ignored, assuming that the system is well above the excitation threshold [2] to delineate the physics of radial propagation. Moreover, $V_S = 2C_d\omega_*\rho_i^2k_0$ and $V_G = C_G\omega_G\rho_i^2k_0$ are, respectively, the linear group velocities of DW sideband and GAM. We note that, V_S and V_G have the same sign for typical tokamak parameters [2, 8], such that the excitation of GAM by DW turbulence is a convective amplification process, ignoring system nonuniformities [13, 20]. In deriving equations (3) and (4), the following frequency and wavenumber matching conditions for resonant decay are applied

$$-\omega_0 + \omega_P - \omega_* + C_d\omega_*k_0^2\rho_i^2 \;=\; 0,$$

$$-\omega_0^2 + \omega_G^2 + C_G\omega_G^2k_0^2\rho_i^2 \;=\; 0,$$

where (ω_0, k_0) for resonant decay can be solved.

Moving into the wave frame by taking $\zeta = \xi - V_c\tau$, with $V_c = (V_S + V_G)/2$, the coupled nonlinear equations, (3) and (4), can then be combined to yield the following equation describing the nonlinear spatial-temporal evolution of the parametrically excited GAM

$$\left(\partial_\tau^2 - V_0^2\partial_\zeta^2\right)\mathscr{E} = \frac{1}{2}k_0^2\Gamma_0^2\mathscr{E} - ik_0\Gamma_0^2\partial_\zeta\mathscr{E}. \tag{5}$$

Here, $V_0 = (V_S - V_G)/2$. Letting $\mathscr{E} = \exp(i\beta\zeta)A(\zeta,\tau)$, with $\beta = k_0\Gamma_0^2/(2V_0^2)$, equation (5) reduces to

$$\left(\partial_\tau^2 - V_0^2\partial_\zeta^2\right)A = \left(\frac{1}{2}k_0^2\Gamma_0^2 + \beta k_0\Gamma_0^2 - \beta^2V_0^2\right)A \equiv \hat{\eta}^2A. \tag{6}$$

Equation (6) can be solved, and yield the following unstable solution

$$A = \hat{A}_0 \exp\left[ik_I\zeta + \sqrt{\eta^2 - k_I^2 V_0^2}\tau\right], \tag{7}$$

in which, k_I is the wavenumber conjugate to ζ at $\tau = 0$, and $\hat{A}_0$ is a constant. Assuming $|V_0\partial_\zeta| \ll |\partial_\tau|$, i.e., convective damping due to FLR effects are higher order corrections to the temporal growing [2, 21], the general solution, (7) can be reduced to the following time asymptotic solution:

$$A \;=\; A_0 \exp\left(\hat{\eta}\tau - \frac{\hat{\eta}\zeta^2}{2V_0^2\tau}\right), \tag{8}$$

in which,

$$A_0 = \hat{A}_0 \exp\left[-\frac{V_0^2\tau}{2\eta}\left(k_I - \frac{i\hat{\eta}\zeta}{V_0^2\tau}\right)^2\right].$$

The time asymptotic solution of GAM electric field, $\mathscr{E}$, is then

$$\mathscr{E} = \mathscr{E}_0 \exp\left(\hat{\eta}\tau + i\beta(\xi - V_c\tau) - \frac{\hat{\eta}}{2V_0^2\tau}(\xi - V_c\tau)^2\right), \tag{9}$$

which has an oscillation (the $i\beta(\xi - V_c\tau)$ term in the exponent) due to the modulation of the pump DW, besides the dominant temporal growth (the $\hat{\eta}\tau$ term in the exponent). At the same time, the envelope is propagating at $V_c = (V_S + V_G)/2$, which is much larger than the linear group velocity of GAM. We note that the validity condition for the time asymptotic solution given in equation (9) is $|\xi - V_c\tau| \ll |V_0\tau|$.

One then have readily from equation (9) that, the nonlinearly excited GAM, is characterized by a nonlinear radial wavevector

$$k_{NL} = k_0 - i\partial_\xi \ln \mathscr{E} = k_0\left(1 + \Gamma_0^2/(2V_0^2)\right), \tag{10}$$

i.e., the wavevector increases with pump DW amplitude, and is larger than that predicted from frequency/wavenumber matching conditions.

The real frequency of the excited GAM, can also be obtained from equation (9),

$$\omega_{NL} = \omega_0 + i\partial_\tau \ln \mathscr{E} = \omega_0 + \frac{k_0\Gamma_0^2 V_c}{2V_0^2}. \tag{11}$$

$\omega_0(k_0)$ can be solved from the matching conditions, which can then be substituted into equation (11), and yield:

$$\begin{aligned}\omega_{NL} \;&=\; \omega_G + \frac{k_0\Gamma_0^2 V_c}{2V_0^2} + \frac{1}{2}C_G\omega_G k_0^2\rho_i^2 \\ &=\; \omega_G + \frac{k_0\Gamma_0^2 V_c}{2V_0^2} + \frac{C_G\omega_G\rho_i^2 k_{NL}^2}{2(1 + \Gamma_0^2/(2V_0^2))^2}.\end{aligned} \tag{12}$$

This is the nonlinear dispersion relation of the parametrically excited GAM. We note that, both V_0 and V_c are proportional to k_0, and thus, the nonlinear frequency shift due to the modulation of DW, $k_0\Gamma_0^2V_c/(2V_0^2)$, is independent of k_0. Thus, finite amplitude DW will increase the frequency of the nonlinearly driven GAM. This may explain the existence of the higher frequency branch of the "dual-GAM" observed in HT-7 tokamak [16]. On the other hand, the coefficient for kinetic dispersiveness, is in fact, decreased by a factor $(1+\Gamma_0^2/(2V_0^2))^2$. The reason why experimental analysis found an "increased" coefficient is that, in the analysis of experimental data, one employed the linear dispersion relation of GAM and used the expression $(\omega_{obs}-\omega_{loc})/(\omega_{loc}k_{obs}^2\rho_i^2)$ to determine the coefficient C_G. Here, the subscript "*obs*" denotes experimental observation, and "*loc*" denotes local continuum frequency of GAM. Since as we have shown in equation (12), "$\omega_{obs}-\omega_{loc}$" contains, besides the kinetic dispersiveness, also the nonlinear frequency increment $k_0\Gamma_0^2V_c/(2V_0^2)$; which, thus, can lead to an over-estimation of the coefficient C_G^{NL} [22].

The coupled GAM and DW sideband wavepacket, propagates at a nonlinear group velocity $V_c=(V_S+V_G)/2$, which is much larger than the linear group velocity of GAM due to $|V_S|\gg|V_G|$ ($|\omega_P|\simeq|\omega_*|\gg|\omega_G|$ for resonant decay). Thus, to interpret the propagation of GAM nonlinearly excited by DW turbulences including DAW, linear theory of KGAM [2, 17] is not adequate, and one must apply the nonlinear theory here. We note also that, while both the real frequency and wavevector of the excited GAM depend on the amplitude of the pump DW, the nonlinear group velocity, is determined by k_0 from matching conditions, and is independent of the pump amplitude. Thus, for the comparison of experimentally observations with analytical theory, the nonlinear group velocity may be a better candidate.

The coupled nonlinear GAM and DW sideband equations, equations (1) and (2), are solved numerically. Here, we fix $C_d=C_G=1$, $\omega_G=0.1$, $\omega_P=\omega_*=1$, and study the coupled nonlinear equations by varying Γ_0. The dependence of the nonlinear wavenumber k_r on pump amplitude is given in Fig. 1, where the dots are the wavenumbers from numerical solution, the diamonds are the obtained from equation (10); and the solid curve is obtained from matching condition. For the parameters we have here, the wavevector solved from matching conditions $k_0=0.32$. We may see from Fig. 1 that, our nonlinear theory fits well with the numerical results; and it reduces to k_0 as Γ_0 approaches 0. The comparison of the numerically measured nonlinear group velocity with our theory, is presented in Fig. 2, where the dots are numerical results and the diamonds are obtained from $V_c=(V_S+V_G)/2$, and V_S and V_G are defined with k_0. We note that, for the parameters we used in numerical solution, $V_S=0.64$, $V_G=0.032$ and $V_c=(V_S+V_G)/2=0.34\gg V_G$. Very good agreement between numerical results and analytical theory ($<3\%$ discrepency) are obtained here, suggesting that experimentally observed radial propagation of GAM must be understood

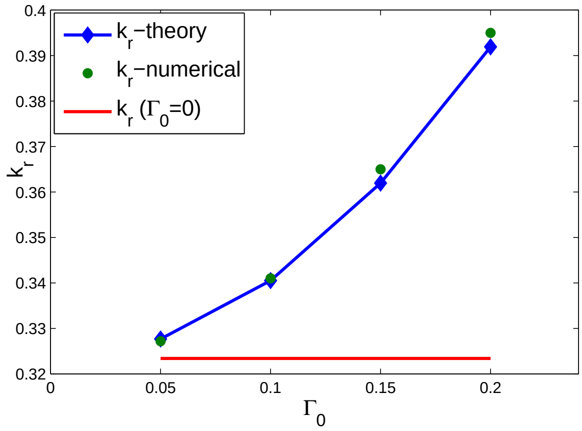

FIG. 1: Nonlinear wavenumber k_r v.s. pump amplitude Γ_0

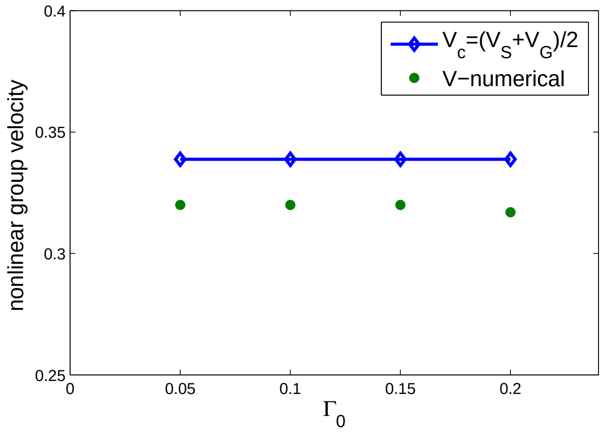

FIG. 2: Nonlinear group velocity v_g v.s. pump amplitude Γ_0

using nonlinear theory.

The nonlinear frequency of GAM is given in Fig. 3, where the dots are numerical results and the diamonds represents ω_{NL} from equation (11). Note that, for the parameters we use here, $\omega_0 = 0.105$, and the nonlinear frequency from numerical solution increases with pump DW amplitude, as predicted by our theory.

In conclusion, the equations describing the spatial/temporal evolution of parametrically coupled GAM and DW are studied both analytically and numerically. It is found that, the parametrically excited GAM propagate at a nonlinear group velocity, which is the mean of the linear group velocities of GAM and DW turbulence, and is much larger than that

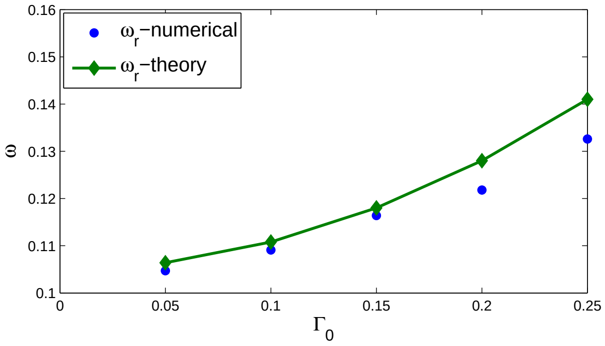

FIG. 3: Nonlinear GAM frequency ω^{NL} v.s. pump amplitude Γ_0

predicted by linear theory of kinetic GAM. The wavevector of the excited GAM, has a quadratic dependence on the amplitude of the constant-amplitude pump DW. On the other hand, the nonlinear group velocity is independent of the pump DW amplitude; suggesting it as a good candidate for the comparison between experiments and analytical theory. Our nonlinear theory, further shows that there is a nonlinear up shift in the GAM frequency. Implications of the present theoretical findings to the HT-7 experimental observations are also discussed. Our results demonstrate that one must include nonlinear effects in order to properly analyze experimental observations of GAM.

Acknowledgments

This work is supported by US DoE GRANT, the ITER-CN under Grants Nos. 2011GB105001, 2013GB104004 and 2013GB111004, the National Science Foundation of China under grant Nos. 11205132, 11203056 and 11235009, and Fundamental Research Fund for Chinese Central Universities.

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